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On the Projective Deformation of Curved Surfaces. I.

By

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Introduction.

In the year 1916 M. G. Fubini has published a memoir on the projective deformation and discussed the condition that two surfaces may be projectively deformable and the relations between such surfaces.[1]

Afterwards a great advance was made by Cartan in the theory of the projective deformation.[2]

He noticed that the surface which admits the projective deformation is exeptional one.[3] In another words for a surface admits a projective deformation (which is not a projective transformation) certain conditions must be satisfied.

Cartan has proved this fact by aid of the theory of the differential equation, we shall express this condition in terms of the quantities which define the surface. This can be expressed in a simple form. And then we will discuss, by means of it, some geometrical characteristic properties of the surface which admits the projective deformation.

§ I. Cartan's conditions that two surfaces may be projectively deformable.

We say two surfaces (S) and (T) are projectively deformable when they are in the following relations:

1 Rendiconti del circol matematico di Palermo. 41, 135, (1916).

2 Annales scientifiques de l'école normale supérieure. (3), 37, 259 (1920).

3 Loc. cil cit.

1° There is a one to one correspondence between the points of the surfaces (S) and (T).

2° If M and P be a pair of the corresponding points respectively on (S) and (T), we can displace (T) in the projective space (that is, apply a projective transformation) so as P coincides with M, and (T) has a contact of the second order with (S) at M.

Let us now confine ourselves to the three dimensions, and start from Cartan's conditions for the projective deformability of two surfaces.

Take the following four points:

1° A point A on the surface (S)

2° Two points A_1 and A_2 on the tangent plane at A of the surface

3° A point A_3 outside of the tangent plane.

We can regard the above four points as the vertices of the moving coordinate tetrahedron.

Let (x), (y), (z), (t), (p) be homogeneous coordinates of the points A, A_1, A_2, A_3, and any point P referring to the fixed axes.

Then the coordinates (w) of the point P referring to the moving axes are given by the equations

$$p_i = w_0 x_i + w_1 y_i + w_2 z_i + w_3 t_i. \qquad i = 0, 1, 2, 3.$$

Referring to the fixed axes, the coordinates of the points A, A_1, A_2, A_3 will generally depend, besides the parameters u and v which define the coordinates of the point on the surface (S), on some other parameters.

When we give these parameters infinitely small variations, the coordinates referred the moving axes of the points (dx), (dy), (dz), (dt) are given by the equations

$$\begin{aligned} dx_i &= w_{00} x_i + w_{01} y_i + w_{02} z_i + w_{03} t_i, \\ dy_i &= w_{10} x_i + w_{11} y_i + w_{12} z_i + w_{13} t_i, \\ dz_i &= w_{20} x_i + w_{21} y_i + w_{22} z_i + w_{23} t_i, \\ dt_i &= w_{30} x_i + w_{31} y_i + w_{32} z_i + w_{33} t_i, \end{aligned} \qquad (i = 0, 1, 2, 3)$$

w_{0i}, w_{1i}, w_{2i}, w_{3i} $(i = 0, 1, 2, 3)$ being required coordinates.

As the points (dx) is evidently on the tangent plane, we have

$$w_{03} = 0.$$

Evidently, w's are Pfaff's expressions.

For a surface (T) is projectively deformable to (S), it is necessary and sufficient that corresponding to A, A_1, A_2, A_3, we can take a point $B(\xi)$ on (T), two points $B_1(\eta)$, $B_2(\varsigma)$ on the tangent plane at B of (T), and a point $B_3(\tau)$ outside the tangent plane so as to satisfy the following system of simultaneous total differential equations

$$\left.\begin{aligned} &W_{01}=w_{01}, \qquad W_{02}=w_{02},\\ &W_{12}=w_{12}, \qquad W_{21}=w_{21},\\ &W_{13}=w_{13}, \qquad W_{23}=w_{23},\\ &W_{00}-w_{00}=W_{11}-w_{11}=W_{22}-w_{22}. \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots (\mathrm{I})$$

where W_{0i}, W_{1i}, W_{2i}, W_{3i} $(i=0, 1, 2, 3)$ are coordinates of the points $(d\xi)$, $(d\eta)$, $(d\zeta)$, $(d\tau)$ respectively referring to the moving coordinate tetrahedron formed by the points $B(\xi)$, $B(\eta)$, $B_2(\varsigma)$, $B_3(\tau)$.

These are the Cartan's conditions.[1]

§ 2. Particularization of the moving coordinate axes.

Let us suppose the surface (T) is projectively deformable to the surface (S), and the parameters u, v are taken so as the corresponding points on (S) and (T) correspond to the same values of u, v, and on the surface (S) the curves u-const. and v-const. are the asymptotic lines.

Then if the surface (S) is not developpable, as showed by Wilzynski, the coordinates of the point on (S) may be regarded as the fundamental system of the solutions of the system of the simultaneous partial differential equations[2]

$$\frac{\partial^2 x}{\partial u^2}=a_1\frac{\partial x}{\partial u}+b_1\frac{\partial x}{\partial v}+c_1 x,$$

$$\frac{\partial^2 x}{\partial u^2}=a_2\frac{\partial x}{\partial u}+b_2\frac{\partial x}{\partial v}+c_2 x,$$

which are transformed by the transformation of the form

$$\bar{x}=\lambda x. \quad \cdots\cdots\cdots\cdots (\mathrm{I})$$

to the equations

$$\left.\begin{aligned} &\frac{\partial^2 x}{\partial u^2}=b\frac{\partial x}{\partial v}+fx\\ &\frac{\partial^2 x}{\partial v^2}=a\frac{\partial x}{\partial u}+gx \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots (\mathrm{II})$$

1 Loc. cit. 272.

2 Traus. Amer. Math. Soc., 8, 244 (1917).

where $b=b_1, \qquad a=a_2.$

The functions, a, b, f, g of u, v are restricted by the conditions of the integrability:

$$\left.\begin{aligned} 2\frac{\partial g}{\partial u} &= \frac{\partial(ab)}{\partial v} + a\frac{\partial b}{\partial v} - \frac{\partial^2 a}{\partial u^2} \\ 2\frac{\partial f}{\partial v} &= \frac{\partial(ab)}{\partial u} + b\frac{\partial a}{\partial u} - \frac{\partial^2 b}{\partial v^2} \\ \frac{\partial^2 f}{\partial v^2} + b\frac{\partial g}{\partial v} + 2g &= \frac{\partial g}{\partial u^2} + a\frac{\partial f}{\partial u} + 2f\frac{\partial a}{\partial u} \end{aligned}\right\} \quad \text{(III)}$$

As the transformation (1) does not alter the surface, there is no loss of generality, if we suppose the coordinates (x) of the point on (S) to satisfy the equations (II).

Let us now take $\left(\frac{\partial x}{\partial u}\right)$ for (y), $\left(\frac{\partial x}{\partial v}\right)$ for (z) and $\left(\frac{\partial^2 x}{\partial u \partial v}\right)$ for (t).

Then from the equation (II), we have

$$d\begin{vmatrix} x^0 & \frac{\partial x_0}{\partial u} & \frac{\partial x_0}{\partial v} & \frac{\partial^2 x_0}{\partial u \partial v} \\ x_1 & \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} & \frac{\partial^2 x_1}{\partial u \partial v} \\ x_2 & \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} & \frac{\partial^2 x_2}{\partial u \partial v} \\ x_3 & \frac{\partial x_3}{\partial u} & \frac{\partial x_3}{\partial v} & \frac{\partial^2 x_3}{\partial u \partial v} \end{vmatrix} = 0,$$

$$\therefore \quad \begin{vmatrix} x & \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial^2 x}{\partial u \partial v} \end{vmatrix} = \text{const.}$$

As we have assumed the surface (S) is not developpable, the value of the above determinant is not zero.

From equation (II), we have

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv,$$

$$d\left(\frac{\partial x}{\partial u}\right) = \frac{\partial^2 x}{\partial u^2}du + \frac{\partial^2 x}{\partial u \partial v}dv = fx\,du + b\frac{\partial x}{\partial v}du + \frac{\partial^2 x}{\partial u \partial v}dv,$$

$$d\left(\frac{\partial x}{\partial v}\right) = \frac{\partial^2 x}{\partial u \partial v}du + \frac{\partial^2 x}{\partial v^2}dv = gx\,dv + a\frac{\partial x}{\partial u}dv + \frac{\partial^2 x}{\partial u \partial v}du,$$

$$d\left(\frac{\partial^2 x}{\partial u \partial v}\right)=\frac{\partial^3 x}{\partial u \partial v}du+\frac{\partial^3 x}{\partial u \partial v^2}dv$$

$$=\left\{(bg+f_v)du+(af+g_u)dv\right\}x$$
$$+\left\{abdu+(a_u+g)dv\right\}\frac{\partial x}{\partial u}+\left\{(b_v+f)du+abdv\right\}\frac{\partial x}{\partial v}.$$

Hence we have

$$w_{00}=w_{11}=w_{22}=w_{33}=0,$$
$$w_{01}=du, \quad w_{02}=dv, \quad w_{03}=0,$$
$$w_{10}=fdu, \quad w_{12}=bdu, \quad w_{13}=dv,$$
$$w_{20}=gdv, \quad w_{21}=adv, \quad w_{23}=du,$$
$$w_{31}=abdu+(a_u+g)dv,$$
$$w_{32}=(b_v+f)du+abdv,$$
$$w_{30}=(bg+f_v)du+(af+g_u)dv.$$

The conditions (I) become

$$\left.\begin{aligned} &W_{01}=W_{23}=du,\\ &W_{02}=W_{13}=dv,\\ &W_{12}=bdu, \qquad W_{21}=adv,\\ &W_{00}=W_{11}=W_{22}. \end{aligned}\right\} \quad \text{.....(IV)}$$

And for the same reason as $w_{03}=0$, we have $W_{03}=0$.

§ 4. Determination of B_1, B_2, B_3.

Equation (IV) is a system of simultaneous total differential equations.

By the theorem of Frobenius[1] the conditions of integrability of (4) are that, denoting with $'$ the bilinear covariant of the original expression, the equations

$$W'_{03}=0$$
$$W'_{01}=W'_{23}=(du)'=0,$$
$$W'_{02}=W'_{13}=(dv)'=0,$$
$$W'_{12}=(bdu)', \qquad W'_{21}=(adv)',$$
$$W'_{00}=W'_{11}=W'_{22}$$

1 Crel. J., **82**, 276, (1877) Crelle journal t. 82 1877 p. 276.

hold as the consequence of (IV).

Now let us consider two expressions of Pfaff w_d, W_d such that

$$w_d = a_0 dx_0 + a_1 dx_1 + \ldots\ldots + a_n dx_n,$$
$$W_d = b_0 dx_0 + b_1 dx_1 + \ldots\ldots + b_n dx_n,$$

and denote with w_δ, W_δ the following expressions

$$w_\delta = a_0 \delta x_0 + a_1 \delta x_1 + \ldots\ldots + a_n \delta x_n,$$
$$W_\delta = b_0 \delta x_0 + b_1 \delta x_1 + \ldots\ldots + b_n \delta x_n,$$

where δx_i means another variation of x_1.

Denote with Cartan the expression $w_d W_\delta - w_\delta W_d$ with the symbol $[wW]$.

Then the bilinear covariant of W_{ij} is given by the following formulas.[1]

$$W'_{ij} = [W_{i0} W_{0j}] + [W_{i1} W_{1j}] + [W_{i2} W_{2j}] + [W_{i3} W_{3j}]. \ldots\ldots (V)$$

By the above theorem, from the equations.

$$W'_{13} = 0,$$
$$W'_{23} = 0,$$

we have

$$[du(W_{33} - W_{22})] = 0,$$
$$[dv(W_{33} - W_{11})] = 0.$$

But $W_{22} = W_{11}$ and du, dv are independent.

Hence we conclude

$$W_{00} = W_{11} = W_{22} = W_{33}.$$

Let us suppose we have taken the coordinates (ξ), (η), (ζ), (τ) of B, B_1, B_2, B_3 respectively so as to satisfy

$$\begin{vmatrix} \xi_0 & \eta_0 & \zeta_0 & \tau_0 \\ \xi_1 & \eta_1 & \zeta_1 & \tau_1 \\ \xi_2 & \eta_2 & \zeta_2 & \tau_2 \\ \xi_3 & \eta_3 & \zeta_3 & \tau_3 \end{vmatrix} = 1,$$

this being always possible.

[1] Cartan: Bull. math. France, 48, 146 (1920) t. XLVIII 1920 p. 146.

Then we have

$$\begin{aligned} 0 &= d\,|\ \xi\ \ \eta\ \ \zeta\ \ \tau\ | \\ &= W_{00}+W_{11}+W_{22}+W_{33} \\ \therefore \quad & W_{00}=W_{11}=W_{22}=W_{33}=0. \end{aligned}$$

From $W'_{00}=0$ follows by the formulas (V)

$$[du\,W_{10}]+[dv\,W_{20}]=0.$$

Hence by the theorem of Cartan,[1] W_{10}, W_{20} must be linear combinations of du and dv such that

$$\begin{aligned} W_{10} &= h\,du+k\,dv, \\ W_{20} &= k\,du+l\,dv, \end{aligned}$$

where h, k, l, are certain functions of the parameters which define the coordinates of the points B, B_1, B_2, B_3.

If $k\neq 0$, we consider B_3+kB as new B_3.

Evidently by this transformation, the expression W_{ij} which appear in the conditions (I) do not alter, but W_{10} and W_{20} become $W_{10}-kW_{13}$ and $W_{20}-kW_{23}$ respectively.

Hence we can reduce W_{10}, W_{20} to the forms

$$\begin{aligned} W_{10} &= \lambda\,du, \\ W_{20} &= \mu\,dv. \end{aligned}$$

Now we have by differentiation

$$d\xi_i=\frac{\partial \xi_i}{\partial u}du+\frac{\partial \xi_i}{\partial v}dv,$$

On the other hand, we have

$$d\xi_i=\eta_i\,du+\zeta_i\,dv.$$

$$\therefore \quad (\eta)=\left(\frac{\partial \xi}{\partial u}\right),$$

$$(\zeta)=\left(\frac{\partial \xi}{\partial v}\right).$$

From the relations

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[1] Loc. cit.

$$W_{10}=\lambda du,$$
$$W_{12}=b du,$$
$$W_{13}=dv,$$
$$W_{11}=0,$$

we have

$$d\left(\frac{\partial\xi_i}{\partial u}\right)=\frac{\partial^2\xi_i}{\partial u^2}du+\frac{\partial^2\xi_i}{\partial u\partial u}dv$$
$$=\left(\lambda\xi_i+b\frac{\partial\xi_i}{\partial v}\right)du+\tau_i dv.$$

$$\therefore\qquad(\tau)=\left(\frac{\partial^2\xi}{\partial u\partial v}\right),$$

$$\frac{\partial^2\xi_i}{\partial u^2}=b\frac{\partial\xi_i}{\partial v}+\lambda\xi_i.$$

From the relations

$$W_{20}=dv,$$
$$W_{21}=a dv,$$
$$W_{22}=0,$$
$$W_{23}=du,$$

we have

$$d\left(\frac{\partial\xi_i}{\partial v}\right)=\frac{\partial^2\xi_i}{\partial u\partial v}du+\frac{\partial^2\xi_i}{\partial v^2}dv$$
$$=\left(\mu\xi_1+a\frac{\partial\xi_1}{\partial u}\right)dv+\frac{\partial^2\xi_9}{\partial u\partial v}du.$$

$$\therefore\qquad\frac{\partial^2\xi_i}{\partial v^2}=a\frac{\partial\xi_i}{\partial v}+\mu\xi_i.$$

Hence we know that (ξ) must be a system of the fundamental solutions of a partial differential equation

$$\left.\begin{aligned}\frac{\partial^2 y}{\partial u^2}&=b\frac{\partial y}{\partial v}+\lambda y,\\ \frac{\partial^2 y}{\partial v^2}&=a\frac{\partial y}{\partial u}+\mu y.\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots\cdots\cdots\ (\mathrm{VI})$$

The conditions of integrability of the system (VI) are given by

$$\left.\begin{aligned} 2\frac{\partial\lambda}{\partial v} &= \frac{\partial(ab)}{\partial u} + b\frac{\partial a}{\partial u} - \frac{\partial^2 b}{\partial v^2}, \\ 2\frac{\partial\mu}{\partial u} &= \frac{\partial(ab)}{\partial v} + a\frac{\partial b}{\partial v} - \frac{\partial^2 a}{\partial u^2}, \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \text{(VII)}$$

$$\frac{\partial^2\lambda}{\partial v^2} + b\frac{\partial\mu}{\partial v} + 2\mu\frac{\partial b}{\partial v} = \frac{\partial^2\mu}{\partial u^2} + a\frac{\partial\lambda}{\partial u} + 2\lambda\frac{\partial a}{\partial u}.$$

We can easily prove by preceding method that (VII) are also the conditions of integrability of the system of simultaneous total differential equations (IV).

Reciprocally, if (ξ) be a system of the fundamental solutions of the system (VI), we can, by taking (ξ) for B, $\left(\frac{\partial\xi}{\partial u}\right)$ for B_1 and $\left(\frac{\partial\xi}{\partial v}\right)$ for B_2 and $\frac{\partial^2\xi}{\partial u\partial v}$ for B_3, easily prove that the surface generated by the point whose coordinates are (ξ) is projectively deformable to (S).

Hence we know that *for two nondeveloppable surface are projectively deformable to each other, it is necessary and sufficient that Wilczynski's invariant a and b have equal values at the corresponding points of the two surfaces.*

If one of a and b are equal to zero, the surface is a ruled surface.

Hence we know that the surface which are projectively deformable to a curved surface are also curved. We will hereafter consider only the curved surface, that is, we assume $a \neq 0$, $b \neq 0$.

§ 4. Condition that a curved surface may admit projective deformation.

From the first and second equations of (III) and (VII), we have

$$\frac{\partial\lambda}{\partial v} = \frac{\partial f}{\partial v},$$

$$\frac{\partial u}{\partial u} = \frac{\partial g}{\partial u},$$

$$\therefore \quad \lambda = f + \{U(u)\}^2,$$

$$\mu = g + \{V(v)\}^2,$$

where $U(u)$ is a function of u only and $V(v)$ is a function of v only.

Substituting these values in the third equations of (III) and (VII), we have

$$U(u)\left\{U(u)\frac{\partial a}{\partial u} + U'(u)a\right\} = V(v)\left\{V(v)\frac{\partial b}{\partial v} + V'(v)b\right\}.$$

Now we have various cases.

Case I. If

$$U(u)=V(v)=0,$$

we have

$$\lambda=f,$$

$$\mu=g.$$

In this case (T) is a mere projective transform of (S).

Case II. If only one of the $U(u)$ and $V(v)$ are equal to zero, e.g., if $V(v)=0$, then we have

$$U(u)\frac{\partial a}{\partial u}+U'(u)a=0,$$

$$\frac{\partial}{\partial u}(Ua)=0.$$

Hence a takes the form

$$a=\frac{fn(v)}{U(u)}=\frac{U(v)}{U(u)}\,(say).$$

Similarly if $U(u)=0$, b takes the form

$$b=\frac{fn(u)}{V(v)}=\frac{U(u)}{V(v)}\,(say).$$

By the transformation of the form

$$\bar{x}=C\sqrt{UV}\,x,\qquad d\bar{u}=U(u)du,\qquad d\bar{v}=V(v)dv,$$

the surface and also the parameter curves do not alter and the system (II) is transformed into another of the same form in which, denoting with $\bar{a}$, $\bar{b}$ the corresponding quantities, we have[1]

$$\bar{a}=\frac{U(u)}{\{V(v)\}^2},\qquad \bar{b}=\frac{V(v)}{\{U(u)\}^2}.$$

Hence in the former case by the transformations

$$\bar{x}=C\sqrt{U}\sqrt[4]{V}\,x\qquad d\bar{u}=Udu,\qquad d\bar{v}=\sqrt{V}dv$$

we have

$$\bar{a}=1.$$

In the latter case by the tranformations

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[1] Wilczynski, loc. cit.

$$\bar{x}=C\sqrt[4]{U}\sqrt{V}x, \qquad d\bar{u}=\sqrt{U}du, \qquad dv=Vd\bar{v},$$

we have

$$b=1.$$

Case III. If $U(u)\neq 0$, $V(v)\neq 0$, we have

$$\frac{U(u)\dfrac{\partial a}{\partial u}+U'(u)a}{V(v)}=\frac{V(v)\dfrac{\partial b}{\partial v}+V'(v)b}{U(u)},$$

$$\frac{\partial}{\partial u}\left(\frac{U(u)a}{V(v)}\right)=\frac{\partial}{\partial v}\left(\frac{V(v)b}{U(u)}\right).$$

Hence we can determine by a quadrature a function p which satisfies the relation

$$a=\frac{V(v)}{U(u)}\frac{\partial p}{\partial v},$$

$$b=\frac{U(u)}{V(v)}\frac{\partial p}{\partial u}.$$

By the transformations

$$\bar{x}=C\sqrt{UV}x, \qquad d\bar{u}=Udu, \qquad d\bar{v}=Vdv,$$

we have

$$\bar{a}=\frac{U(u)}{\{V(v)\}^2}a, \qquad \bar{b}=\frac{V(v)}{\{U(u)\}^2}b.$$

But

$$\frac{\partial p}{\partial v}=\frac{\partial p}{\partial \bar{v}}V(v), \qquad \frac{\partial p}{\partial u}=\frac{\partial p}{\partial \bar{u}}U(u).$$

Hence, we have

$$\bar{a}=\frac{\partial p}{\partial \bar{v}}, \qquad \bar{b}=\frac{\partial p}{\partial \bar{u}}.$$

Therefore, we know that *for the curved surfaces which admit the projective deformation, Wilczynski's invariants, a, b must be reduced so as to satisfy one of the relations.*

(1) $a=I.$

(2) $b=I.$

(3) $\dfrac{\partial a}{\partial u}=\dfrac{\partial b}{\partial v}.$

Reciprocally we will prove in the next paragraph that if a surface satisfies one of the above conditions, there are at least ∞^1 surfaces which are projectively deformable to the surface (S).

Therefore there are three kinds of surfaces which admit the projective transformation. But the surface where $a=1$ and the surface where $b=1$ are of the same kind, for by the interchange of parameters u, v, we can reduce one into the other. We shall call the surface of this kind the surface of the second kind which admits the projective transformation and the surface where $a_u=b_v$ the surface of the first kind which admits the projective transformation.

§ 5. All the surfaces which are projectively deformable to a given surface.

Let us suppose that we have performed the transformations explained in the preceding paragraph.

First, we consider the case where

$$\frac{\partial a}{\partial u}=\frac{\partial b}{\partial v},$$

or

$$a=-\frac{\partial p}{\partial v}, \qquad b=\frac{\partial p}{\partial u}.$$

If the equations of the given surface be

$$\frac{\partial^2 x}{\partial u^2}=b\frac{\partial x}{\partial v}+fx,$$

$$\frac{\partial^2 x}{\partial v^2}=a\frac{\partial x}{\partial u}+gx,$$

where a, b, f and g are restricted by the conditions of the integrability, then all the surfaces which are projectively deformable to it, if they exist, are given by

$$\frac{\partial^2 x}{\partial u^2}=b\frac{\partial x}{\partial v}+\lambda x,$$

$$\frac{\partial^2 x}{\partial v^2}=a\frac{\partial x}{\partial u}+\mu x,$$

where λ, u must satisfy the conditions of integrability

$$\frac{\partial \lambda}{\partial v}=\frac{\partial f}{\partial v},$$

$$\frac{\partial \mu}{\partial u}=\frac{\partial g}{\partial u},$$

$$\frac{\partial^2\mu}{\partial u^2}+a\frac{\partial\lambda}{\partial u}+2\lambda\frac{\partial a}{\partial u}=\frac{\partial^2\lambda}{\partial v^2}+b\frac{\partial\mu}{\partial v}+2\mu\frac{\partial b}{\partial v}.$$

If we assume

$$\lambda=f+U(u),$$
$$u=g+V(v),$$

the functions $U(u)$, $V(v)$ must satisfy the equation

$$aU'(u)+2U\frac{\partial a}{\partial u}=bV'(v)+2V\frac{\partial b}{\partial v},$$

or
$$2(U-V)\frac{\partial^2 p}{\partial u\partial v}-V'(v)\frac{\partial p}{\partial u}+U'(u)\frac{\partial p}{\partial v}=0. \quad \text{.....(1)}$$

This equation is satisfied by

$$U=V=k. \qquad k: \text{ const.}$$

Hence the surfaces which are given by

$$\left.\begin{aligned}\frac{\partial^2 x}{\partial u^2}&=b\frac{\partial x}{\partial v}+(f+k)x,\\ -\frac{\partial^2 x}{\partial v^2}&=a\frac{\partial x}{\partial v}+(g+k)x,\end{aligned}\right\}\quad \text{.....(2)}$$

are projectively deformable to the given surface.

By Wilczynski's theory,[1] all the surfaces which correspond to the same value of k are projective transforms of a particular one, and moreover we can easily see that two surfaces which correspond to the different values of k can not be projective transforms of the same surface.

Therefore, if we regard the surfaces which may be derived by the projective transformation from the same surface as all the same, the equation (1) gives ∞^1 surfaces which are projectively deformable to the given surface.

If there be other pairs of solutions of $U(u)$ and $V(v)$, p must be a solution of the partial differential equation of the form (1).

If p be a particular solution of

$$2\Big(\alpha(u)-\beta(v)\Big)\frac{\partial^2 p}{\partial u\partial v}-\beta'(v)\frac{\partial p}{\partial u}+\alpha'(v)\frac{\partial p}{\partial v}=0,$$

[1] Loc. cit.

$$U(u)=k\alpha(u)+k_1, \qquad V(v)=k_1\beta(v)+k_1, \qquad k,\ k_1 : \text{const.}$$

are evidently solutions of (1).

If these exist a further pair of solutions of $U(u)$ and $V(v)$, p must satisfy the two partial differential equations

$$2\Big(\alpha(u)-\beta(v)\Big)\frac{\partial^2 p}{\partial u\partial v}-\beta'(v)\frac{\partial p}{\partial u}+\alpha'(u)\frac{\partial p}{\partial v}=0,$$

$$2\Big(U(u)-V(v)\Big)\frac{\partial^2 p}{\partial u\partial v}-V'(v)\frac{\partial p}{\partial u}+U'(u)\frac{\partial p}{\partial v}=0,$$

where

$$U(u)\neq k\alpha(u)+k_1, \qquad V(v)\neq k_1\beta(v)+k_1.$$

By the transformotion

$$x=\alpha(u), \qquad \eta=\beta(v),$$

the above equation is transformed to

$$\left.\begin{aligned} &2(x-y)\frac{\partial^2 p}{\partial x\partial y}-\frac{\partial p}{\partial x}+\frac{\partial p}{\partial y}=0,\\ &2\Big(X(x)-Y(y)\Big)\frac{\partial^2 p}{\partial x\partial y}-Y'(x)\frac{\partial p}{\partial x}+X'(x)\frac{\partial p}{\partial y}=0, \end{aligned}\right\}\ \cdots\cdots\cdots(3)$$

where

$$X(x)\equiv U(u), \qquad Y(y)\equiv V(v),$$
$$X(x)\neq kx+k_1, \qquad Y(y)\neq ky+k_1.$$

From the equations (3), we have

$$-\Big(X-Y-Y'(x-y)\Big)\frac{\partial p}{\partial x}+\Big(X-Y-X'(x-\eta)\Big)\frac{\partial p}{\partial y}=0. \quad \cdots(4)$$

Consider the function

$$q=2\int X dx-2\int Y dy-(X+Y)(x-y).$$

Then the equation (4) is equivalent to

$$\frac{\partial(p,\ q)}{\partial(x,\ y)}=0.$$

$$\therefore \qquad p=F(q).$$

Substituting this in the first equation of (3), we have

$$2(x-y)F''(q)\Big\{X-Y-X'(x-\eta)\Big\}\Big\{X-Y-Y'(x-\eta)\Big\}$$
$$+3F'(q)(x-y)(X'-Y')=0. \quad \ldots\ldots\ldots\ldots(5)$$

As x and y are independent $x-y$ is not always zero. If one of the functions $X-Y-X'(x-y)$, and $X-Y-Y'(x-y)$ vanish identically, by the relation (4) the other must vanish, for

$$\frac{\partial p}{\partial x}=\frac{a}{a'(u)}\neq 0,$$
$$\frac{\partial p}{\partial y}=\frac{b}{\beta'(v)}\neq 0.$$

And if both $X-Y-X'(x-y)=0$ and $X-Y-Y'(x-y)=0$ vanish identically, we can easily see that

$$X=kx+k,$$
$$Y=ky+k.$$

This contradicts our assumption.
Hence,

$$X-Y-X'(x-y)\neq 0,$$
$$X-Y-Y'(x-y)\neq 0.$$

Therefore, from (5) we have

$$\frac{2}{3}\frac{F''(q)}{F'(q)}\Big\{X-Y-Y'(x-y)\Big\}=-\frac{X'-Y'}{X-Y-X'(x-y)},$$
$$\frac{2}{3}\frac{F''(q)}{F'(q)}\Big\{X-Y-X'(x-q)\Big\}=-\frac{X'-Y'}{X-Y-X'(x-y)},$$

or

$$\frac{2}{3}\frac{\partial}{\partial y}\log F'(q)=-\frac{\partial}{\partial y}\log\Big(X-Y-Y'(x-y)\Big),$$
$$\frac{2}{3}\frac{\partial}{\partial x}\log F'(q)=-\frac{\partial}{\partial x}\log\Big(X-Y-Y'(x-y)\Big).$$

Integrating we have

$$\Big\{F'(q)\Big\}^{\frac{2}{3}}=\frac{\xi(x)}{X-Y-X'(x-y)}=\frac{\eta(y)}{X-Y-Y'(x-y)} \quad \ldots\ldots\ldots (6)$$

From the equation

$$0=\frac{\partial^2}{\partial x\partial y}\log\left(\frac{\xi(x)}{\eta(y)}\right)=\frac{\partial^2}{\partial x\partial y}\log\left(\frac{X-Y-X'(x-y)}{X-Y-Y'(x-y)}\right),$$

we can easily deduce

$$\xi(x)=-kX''^{\frac{1}{3}},\qquad \eta(y)=+kY''^{\frac{1}{3}}. \qquad k:\text{const.}$$

If either X'' and Y'' be zero, $F'(q)=0$, accordingly $a=b=0$.
Hence $X''\neq 0,\quad Y''\neq 0.$
Substituting these in (6), we have

$$\frac{X-Y}{x-y}=\frac{X'Y''^{\frac{1}{3}}+Y'X''^{\frac{1}{3}}}{X''^{\frac{1}{3}}+Y''^{\frac{1}{3}}}=f(x,\ y),$$

$$X-Y=f(x,\ y)\ (x-y),$$

$$X'=f(x,\ y)+\frac{\partial f}{\partial x}(x-y),$$

$$Y'=f(x,\ y)-\frac{\partial f}{\partial y}(x-y),$$

$$\frac{\partial f}{\partial x}Y''^{\frac{1}{3}}-X''^{\frac{1}{3}}\frac{\partial f}{\partial y}=0. \qquad \text{.....(7)}$$

Consider the function

$$w=\int X''^{\frac{1}{3}}dx+\int Y^{\frac{1}{3}}dy.$$

Then (7) is equivalent to

$$\frac{\partial(f,\ w)}{\partial(x,\ y)}=0.$$

$$\therefore\quad f=\varphi(w).$$

$$X'=\varphi(w)+\varphi'X''^{\frac{1}{3}}(x-y).$$

Differentiating the above equation by y, we have

$$\frac{X''^{\frac{1}{3}}-Y''^{\frac{1}{3}}}{(x-y)\,X''^{\frac{1}{3}}\,Y''^{\frac{1}{3}}}=\frac{\varphi''(w)}{\varphi'(w)}=fn(w).$$

From the equation

$$\frac{\partial(fn(w)\,w)}{\partial(x,y)}=0,$$

we have

$$\frac{1}{3}(x-y)\left(X''^{-\frac{5}{3}}X'''+Y''^{-\frac{5}{3}}Y'''\right)+X''^{-\frac{2}{3}}-Y''^{-\frac{2}{3}}=0. \quad \ldots\ldots(8)$$

Differentiate the above equation by x and then by y, then we have

$$\frac{d}{dy}(Y''^{-\frac{5}{3}}Y''')-\frac{d}{dx}(X''^{-\frac{5}{3}}X''')=0.$$

$$\therefore \qquad \frac{d}{dy}(Y''^{-\frac{5}{3}}Y''')=-3c,$$

c : const.

$$\frac{d}{dx}(X''^{-\frac{5}{3}}X''')=-3c.$$

or

$$X''^{-\frac{5}{3}}X'''=-3(cx+c_1),$$

$$Y''^{-\frac{5}{3}}Y'''=-3(cy+D_1),$$

$$Y''^{-\frac{2}{3}}=cx^2+2c_1x+c_2,$$

$$Y''^{-\frac{2}{3}}=cy^2+2D_1y+D_2.$$

By the equation (8), we know

$$c_1=D_1,$$
$$c_2=D_2.$$

Therefore

$$X'''^{\frac{1}{3}}=\frac{\varepsilon}{\sqrt{cx^2+2c_1x+c_2}},$$

$\varepsilon=+1$ or -1.

$$Y'''^{\frac{1}{3}}=\frac{\delta}{\sqrt{cy^2+2c_1y+c_2}}.$$

$\delta=+1$ or -1.

First, we assume $c\,c_2-c_1^2\neq 0$.

Then we have

$$X'=\frac{\varepsilon}{c\,c_2-c_1^2}\left(\frac{cx+c_1}{\sqrt{cx^2+2c\,x+c_2}}+c_3\right)$$

$$Y' = \frac{\delta}{c\,c_2 - c_1^2}\left(\frac{c\,y + c_4}{\sqrt{cy^2 + 2c_1 y + c_2}} + D_3\right),$$

$$X = \frac{\delta}{c\,c_2 - c_2^2}\left(\sqrt{cx^2 + 2c_1 x + c_2} + c_3 x + c_4\right),$$

$$Y = \frac{\varepsilon}{c\,c_2 - c_1^2}\left(\sqrt{cy^2 + 2c_1 y + c_2} + D_3 y + D_4\right).$$

By the equation (6), we know

$$\varepsilon = \delta,$$
$$c_3 = D_3,$$
$$c_4 = D_4.$$

Therefore

$$X - Y - X'(x-y) = \frac{\varepsilon}{cc_2 - c_1^2}\left(-\sqrt{cy^2 + 2c_1 y + c_2} + \frac{cxy + c_1(x+y) + c_2}{\sqrt{cx^2 + 2c_1 x + c_2}}\right),$$

$$X - Y - Y'(x-y) = \frac{\varepsilon}{cc_2 - c_1^2}\left(\sqrt{cx^2 + 2c_1 x + c_2} - \frac{cxy + c_1(x+y) + c_2}{\sqrt{cy_2 + 2c_1 y + c_2}}\right),$$

$$\left(F'(q)\right)^{-\frac{2}{3}} =$$

$$\frac{K}{cc_2 - c_1^2}\left[\sqrt{cy^2 + 2c_1 y + c_2}\,\sqrt{cx^2 + 2c_1 x + c_2} - \{cxy + c_1(x+y) + c_2\}\right].$$

$$p = F(q) = K \log \frac{\sqrt{\sqrt{\xi^2 + a} - \xi} - \sqrt{\sqrt{\eta^2 + a} - \eta}}{\sqrt{\sqrt{\xi^2 + a} - \xi} + \sqrt{\sqrt{\eta^2 + a} - \eta}} \qquad \ldots\ldots\ldots\ldots(9)$$

where

$$\xi = cx + c_1, \qquad \eta = cy + c_1,$$
$$a = cc_2 - c_1^2,$$
K : certain constant.

Similarly when $cc_2 - c_1^2 = 0$, we have

$$X = \frac{\varepsilon}{2\sqrt{c}}\,\frac{1}{cx + c_1}\,c_3 x + c_4,$$

$$Y = \frac{\varepsilon}{2\sqrt{c}}\,\frac{1}{cy + c_1} + c_3 y + c_4 \qquad \varepsilon = +1, \text{ or } -1.$$

$$p = K \log \frac{\sqrt{cx + c_1} - \sqrt{cy + c_1}}{\sqrt{cx + c_1} + \sqrt{cy + c_1}}. \qquad K : \text{const.} \ \ldots\ldots\ldots\ldots(9')$$

Therefore we know that when there are the following relations between a and b,

$$a = \frac{\partial p}{\partial v} \quad ; \qquad b = \frac{\partial p}{\partial u},$$

(1°) if p does not satisfy the partial differential equation of the form

$$2\Big(\alpha(u)-\beta(v)\Big)\frac{\partial^2 p}{\partial u\partial v}-\beta'(v)\frac{\partial p}{\partial u}+\alpha'(u)\frac{\partial p}{\partial u}=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

there are only ∞^1 surfaces which are projectively deformable to the given surface and they are given by the equations

$$\frac{\partial^2 x}{\partial u^2}=b\frac{\partial x}{\partial v}+(f+k)x,$$

$$\frac{\partial^2 x}{\partial v^2}=a\frac{\partial x}{\partial u}+(g+k)x,$$

(2°) if p satisfies the equations (10) and does not satisfy another partial differential equation of the same form, but not identical with it, there are ∞^2 surfaces which are projectively deformable to the given surface and they are given by the equations

$$\frac{\partial^2 x}{\partial u^2}=b\frac{\partial x}{\partial v}+\Big(f+k\alpha(u)+k_1\Big),$$

$$\frac{\partial^2 x}{\partial v^2}=a\frac{\partial x}{\partial u}+\Big(g+k\beta(v)+k_1\Big),$$

(3°) if p satisfies the following two partial diffential equations

$$2\Big(\alpha(u)-\beta(v)\Big)\frac{\partial^2 p}{\partial u\partial v}-\beta'(v)\frac{\partial p}{\partial u}+\alpha'(u)\frac{\partial p}{\partial r}=0,$$

$$2\Big(U(u)-V(v)\Big)V'(v)\frac{\partial^2 p}{\partial u}+U'(u)\frac{\partial p}{\partial v}=0,$$

we must have

$$\begin{cases} p=K\log\dfrac{\sqrt{\sqrt{c^2\alpha^2+2cc_1+cc_2}-(c\alpha+c_1)}-\sqrt{\sqrt{c^2\beta+2cc_1\beta+cc_2}-(c\alpha+c_1)}}{\sqrt{\sqrt{c^2\alpha^2+2c_1c\alpha+c_2c}-(c\alpha+c_1)}+\sqrt{\sqrt{c^2\beta+2a_1c_1\beta+c_2c}-(c\alpha+c_1)}}, \\ U(u)=\dfrac{\pm 1}{cc_2-c_1^2}\Big(\sqrt{c\alpha+2c_1\alpha+c_2}+c_3\alpha+c_4\Big), \\ V(v)=\dfrac{\pm 1}{cc_2-c_1^2}\Big(\sqrt{c\beta+2c_1\beta+c_2}+c_3\beta+c_4\Big), \\ \text{where}\quad cc_2-c_1^2\neq 0 \end{cases}$$

or

$$\begin{cases} p = K\log\dfrac{\sqrt{c\alpha+c_1}-\sqrt{c\beta+c_1}}{\sqrt{c\alpha+c_1}+\sqrt{c\beta+c_1}}, \\ U(u) = \dfrac{\pm 1}{2\sqrt{c}}\dfrac{1}{c\alpha+c_1} + c_3\alpha + c_4, \\ V(v) = \dfrac{\pm 1}{2\sqrt{c}}\dfrac{1}{c\beta+c_1} + c_3\beta + c_4, \end{cases}$$

and all the surfaces which are projectively deformable to the given surfaces are given by the equations

$$\begin{cases} \dfrac{\partial^2 x}{\partial u^2} = b\dfrac{\partial x}{\partial v} + \left(f + k\sqrt{c\alpha^2+2c_1\alpha+c_2} + k_1\alpha + k_3\right), \\ \dfrac{\partial^2 x}{\partial v^2} = a\dfrac{\partial x}{\partial u} + \left(g + k\sqrt{c\beta^2+2c_1\beta+c_2} + k_1\beta + k_3\right), \end{cases}$$

or

$$\begin{cases} \dfrac{\partial^2 x}{\partial u^2} = b\dfrac{\partial x}{\partial v} + \left(f + \dfrac{k}{c\alpha+c_1} + k_1\alpha + k_3\right), \\ \dfrac{\partial^2 x}{\partial v^2} = a\dfrac{\partial x}{\partial u} + \left(g + \dfrac{k}{c\beta+c_1} + k_1\alpha + k_3\right), \end{cases}$$ respectively

where k, k_1, k_2 are arbitrarly constants.

Next, we consider the case where

$$a = 1.$$

In this case the equations of the given surface are

$$\frac{\partial^2 x}{\partial u^2} = b\frac{\partial x}{\partial v} + fx,$$

$$\frac{\partial^2 x}{\partial v^2} = \frac{\partial x}{\partial u} + gx.$$

Therefore, if we put

$$\lambda = f + U(u),$$
$$\mu = g + V(v).$$

U, V must satisfy the equation

$$U'(u) = bV'(v) + 2V\frac{\partial b}{\partial v}. \qquad \text{.....(4)}$$

This equation is satisfied by

$$U(u) = k,$$
$$V(v) = 0.$$

If there is another pair of solutions of U and V, b must satisfy the partial differential equation of the form (4).
If b satisfies the equation

$$\alpha'(u) = b\beta'(v) + 2\beta\frac{\partial b}{\partial v}, \quad \text{.....(5)}$$

Then $U = k\alpha + k_1,$
$V = k_1\beta$ $k,\ k_1$: const.

are solutions of (4).

If (4) has a further pair solutions, b must satisfy the two partial differential equations

$$\alpha'(u) = b\beta'(v) + 2\beta\frac{\partial b}{\partial v},$$

$$V'(u) = bV'(v) + 2V\frac{vb}{\partial v},$$

from which we can easily deduce

$$b = \frac{\alpha'\left(\int\frac{dv}{\sqrt{\beta}} + c\right)}{2\sqrt{\beta}},$$

$$U(u) = k_1\alpha(u) + k_2,$$

$$V(v) = \frac{k_1\beta(v)}{\left(\int\frac{dv}{\sqrt{\beta}} + c\right)^2}.$$

Therefore, we know that, when we have

$$a = 1,$$

(1°) if b does not satisfy the partial differential equation of the form

$$\alpha'(u) = b\beta'(v) + 2\beta(v)\frac{\partial b}{\partial v},$$

there are only ∞^1 surfaces which are projectively deformable to the given surface, and they are given by the equations

$$\frac{\partial^2 x}{\partial u_2} = b\frac{\partial x}{\partial v} + (f + k)x,$$

$$\frac{\partial^2 x}{\partial v^2} = \frac{\partial x}{\partial u} + gx,$$

(2°) if b satisfies the above partial differential equation, but not equation of the same form which is not identical with it, there are ∞^2 surfaces which are projectively deformable to the given surface and they are given by the equations

$$\frac{\partial^2 x}{\partial u^2} = b\frac{\partial x}{\partial v} + (f + k\alpha + k_1)x,$$

$$\frac{\partial^2 x}{\partial v^2} = \frac{\partial x}{\partial u} + (g + k\beta)x,$$

(3)° if b satisfies the following two partial differential equations

$$\alpha'(u) = b\beta'(v) + 2\beta\frac{\partial b}{\partial v},$$

$$U'(u) = bV'(v) + 2V\frac{\partial b}{\partial v},$$

we must have

$$b = \frac{\alpha'\left(\int \frac{dv}{\sqrt{\beta}} + c\right)}{2\sqrt{\beta}},$$

$$U(u) = k_1\alpha(u) + k_2,$$

$$V(v) = \frac{k\beta(v)}{\left(\int \frac{dv}{\sqrt{\beta}} + c\right)^2} + k_1\beta(v),$$

and all the surfaces which are projectively deformable to the given surface are given by the equations

$$\frac{\partial^2 x}{\partial u^2} = b\frac{\partial x}{\partial v} + (f + k_1\alpha + k_2)x,$$

$$\frac{\partial^2 x}{\partial v^2} = \frac{\partial x}{\partial u} + \left(g + \frac{k\beta}{\left(\int \frac{dv}{\sqrt{\beta}} + c\right)^2} + k_1\beta\right),$$

which are of the manifoldness ∞^3.

§ 6. .Geometrical interpretation.

We will now investigate the geometrical properties which are peculiar to the surface which admits projective deformation.

Let us begin from the case where

$$\frac{\partial a}{\partial u} = \frac{\partial b}{\partial v}.$$

In this case the conjugate net of the curves

$$du^2 - dv^2 = 0,$$

has very interesting properties.

We shall call with Carton this conjugate net of the projective deformation.

In the subsequent discussion we suppose the equations of the surface are

$$\frac{\partial^2 x}{\partial u^2} = b\frac{\partial x}{\partial v} + fx,$$

$$\frac{\partial^2 x}{\partial v^2} = a\frac{\partial x}{\partial u} + gx.$$

Recently Green has published a Memoire on reciprocal congruence.

Let us denote by C_u a curve $v = \text{const.}$, by C_v a curve $u = \text{const.}$, by R_u the ruled surface formed by the tangents to the curve C_u at the points of a fixed curve C_v and by R_v the similar parametric ruled surfaces formed by tangents to the curve C_v.

The congruence formed by the line l which connects two points

$$(\rho) = \left(\frac{\partial x}{\partial u}\right) - \beta(x), \qquad (\sigma) = \left(\frac{\partial x}{\partial v}\right) - \alpha(x),$$

(where α, β are functions of u, v)
are called the congruence I.

The two tangent planes to R_u and R_v at (ρ) and (σ) respectively intersect at the line l' which connects (x) and the point (z)

$$(z) = \frac{\partial^2 x}{\partial u \partial v} = \alpha\left(\frac{\partial x}{\partial u}\right) - \beta\left(\frac{\partial x}{\partial v}\right).$$

Reciprocally the plane determined by (z) and the tangent of C_u at (x) touches R_u at (ρ), and the plane determined by (z) and the tangent of C_v at (α) touches R_v at (σ).

We say one of the lines l, l' is the reciprocal line of the other.

The congruence formed by l' is called the congruence I', and we speak of the congruences I and I' as reciprocal congruences.

Green has proved that when and only when

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[1] Trans. Amer. Math. Soc., **20**, 79–153, (1919).

$$\frac{\partial \alpha}{\partial u} = \frac{\partial \beta}{\partial v},$$

the developpables of the congruence I' correspond to a conjugate net of the surface and the developpables of the congruence I'' cut the surface in a conjugate net.

For the subsequent use, we shall give a short note on the directrix congruence, ray congruence, and axis congruence.

(1) Directrix congruence.[1]

The two linear complexes which osculate at a point (x) the two asymptotics passing through (x) have in common a linear congruence with directrices d and d'.

The directrix of the first kind, d, lies in the tangent plane of the surface, and connects two points

$$\left(\frac{\partial x}{\partial u}\right) - \frac{a_u}{2a}(x), \qquad \left(\frac{\partial x}{\partial v}\right) - \frac{b_v}{2b}(x).$$

The directrix of the second kind, d'', passes through (x) and through the point

$$\left(\frac{\partial^2 x}{\partial u \partial v}\right) - \frac{b_v}{2b}\left(\frac{\partial x}{\partial u}\right) - \frac{a_u}{2a}\left(\frac{\partial x}{\partial v}\right).$$

The congruences formed by d and d'' are called the congruences of the first and second kind respectively. They are evidently reciprocal congruences.

(2) Ray congruence.[2]

Wilczynski has called the line joining the minus first and first Darboux-Laplace transforms of the point (x) with respect to a conjugate net the ray of the point (x), and the congruence formed by the ray the ray congruence of the conjugate net.

(3) Axis congruence.

Wilczynski has called the intersection of the osculating planes of the two curves of a conjugate net which meet at (x), the axis of the point (x) with respect to that conjugate net, and the congruence formed by the axis the axis congruence.

We shall call the reciprocal line of the axis as the reciprocal axis.

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1 Wilczynski, projective Trans. Amer. Math. Soc., **9**, 79–120, (1908).

2 Trans. Amer. Math. Soc., **16**, 311–327, (1915).

If

$$\mu\, du^2 - dv^2 = 0.$$

be the differential equation of a conjugate net, the ray of the point (x) with respect to that conjugate net is the line which joins the two points

$$\left(\frac{\partial x}{\partial u}\right) + \frac{1}{4\mu}\left(2a\,\mu^2 + \frac{\partial \mu}{\partial u}\right)(x), \qquad \left(\frac{\partial x}{\partial v}\right) + \frac{1}{4\mu}\left(2b - \frac{\partial \mu}{\mu v}\right)(x),$$

the axis is the line which passes through (x) and through the point

$$\left(\frac{\partial^2 x}{\partial u\, \partial v}\right) - \frac{1}{4\mu}\left(2b + \frac{\partial \mu}{\partial v}\right)\left(\frac{\partial x}{\partial u}\right) - \frac{1}{4\mu}\left(2a\mu^2 - \frac{\partial \mu}{\partial u}\right)\left(\frac{\partial x}{\partial v}\right),$$

and the reciprocal axis is the line which connects the points

$$\left(\frac{\partial x}{\partial u}\right) - \frac{1}{4\mu}\left(2a\mu^2 - \frac{\partial \mu}{\partial u}\right)(x), \qquad \left(\frac{\partial x}{\partial v}\right) - \frac{1}{4\mu}\left(2b + \frac{\partial \mu}{\partial v}\right)(x).$$

For the conjugate net of the projective deformation, we have

$$\mu = 1$$

Therefore the ray of (x) with respect to that net is the line which connects the two points

$$\left(\frac{\partial x}{\partial u}\right) + \frac{a}{2}(x), \qquad \left(\frac{\partial x}{\partial v}\right) + \frac{b}{2}(x)$$

and the reciprocal axis of (x) with respect to that net is the line which connects the two points

$$\left(\frac{\partial x}{\partial u}\right) - \frac{a}{2}(x), \qquad \left(\frac{\partial x}{\partial v}\right) - \frac{b}{2}(x).$$

The line which is *a* harmonic conjugate of the line which joins (x) and the intersection of the reciprocal axis and the ray of (x) with respect to the ray and the reciprocal axis, passes through the points

$$\left(\frac{\partial x}{\partial u}\right), \qquad \left(\frac{\partial x}{\partial v}\right).$$

The developpables of the congruence formed by this line evidently correspond to *a* conjugate net.

Now for the surface of the first kind which admits the projective deformation, we have

$$\frac{\partial a}{\partial u} = \frac{\partial b}{\partial v}.$$

Therefore, in this case the intersection of the said line and the directrix of the first kind is

$$a\left(\frac{\partial x}{\partial u}\right) - b\left(\frac{\partial x}{\partial v}\right).$$

The intersection of the ray and the reciprocal axis with respect to the conjugate net of the projective deformation is

$$b\left(\frac{\partial x}{\partial u}\right) - a\left(\frac{\partial x}{\partial v}\right).$$

Therefore, we know that on the surface of the first kind which admits the projective deformation, there is a conjugate net which has the following properties.

1° Denote by l_1 and l_2 the ray and reciprocal axis of (x) with respect to that conjugate net respectively, by l_3 the line which joins (x) and the intersection of l_1 and l_2, and by l_4 the fourth harmonic line of l_1, l_2, l_3.

Then the developables of the congruence formed by l_4 correspond to a conjugate net.

2° Denote by P_1 and P_2 the points where the asymptotic tangents meet l_4, by P_3 the intersection of l_1 and l_2, by P_4 the intersection of the directrix of the first kind and l_4, and by D_1 and D_2, the points where the tangents at (x) to that conjugate net meet l_4.

Then P_3, P_4 are separated harmonically by D_1 and D_2.

Or we may express it in the following manner:—

The points P_1, P_2 and P_3, P_4 form two pairs of involution of which D_2 and D_1 are double points.

Reciprocally, if there be a conjugate net which has the said properties on a surface, that surface admits the projective deformation and is of the first species, and that conjugate net is that of the projective deformation.

Let

$$\mu\, du^2 - dv^2 = 0.$$

be the differential equation of that conjugate net.

Then the line l_4 with respect to that conjugate net connects the two points

$$\left(\frac{\partial x}{\partial u}\right)+\frac{1}{4\mu}\frac{\partial \mu}{\partial u}(x), \qquad \left(\frac{\partial x}{\partial v}\right)-\frac{1}{4\mu}\frac{\partial \mu}{\partial v}(x).$$

As the developpables of the congruence of l_3 correspond to a conjugate net, we have

$$-\frac{\partial}{\partial u}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial v}\right)=\frac{\partial}{\partial v}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial u}\right),$$

$$\text{or} \qquad \frac{\partial^2}{\partial u \partial v}\log \mu = 0,$$

$$\mu=\frac{\{U(u)\}^2}{\{V(v)\}^2}.$$

where $U(u)$ is a function of u only, $V(v)$ is a function of v only.

Perform the transformation

$$\bar{x}=c\sqrt{UV}, \qquad d\bar{\xi}=Udu, \qquad d\eta=Vdv.$$

which transforms the differential equation of the above conjugate net into

$$d\bar{\xi}^2-d\eta^2=0.$$

Let

$$\frac{\partial^2 \bar{x}}{\partial \bar{\xi}^2}=\bar{b}\frac{\partial \bar{x}}{\partial \eta}+\bar{f}\,\bar{x}=0$$

$$\frac{\partial^2 \bar{x}}{\partial \eta^2}=\bar{a}\frac{\partial \bar{x}}{\partial \bar{\xi}}+\bar{g}\,\bar{x}=0$$

be the transformed equations of the surface.

Then by the second property of the above conjugate net, we have

$$\frac{\partial \bar{a}}{\partial \bar{\xi}}=\frac{\partial \bar{b}}{\partial \eta},$$

that is, the surface is that of the first kind which admits the projective deformation.

Next, we consider the case where $a=I$.

In this case we consider the canonical (or scroll) congruence which is due to Sullivan and Green.

The canonical edge of the first kind is the line which connects the points

$$\left(\frac{\partial x}{\partial u}\right)+\frac{b_u}{4b}(x), \qquad \left(\frac{\partial x}{\partial v}\right)+\frac{a_v}{4a}(x),$$

and that of the second kind is the line which passes through (x) and through the point

$$\left(\frac{\partial^2 x}{\partial u \partial v}\right) + \frac{a_v}{4a}\left(\frac{\partial x}{\partial u}\right) + \frac{b_u}{4b}\left(\frac{\partial x}{\partial v}\right).$$

Now we have

$$\frac{\partial a}{\partial u} = 0, \qquad \frac{\partial a}{\partial v} = 0.$$

Therefore the intersection of the tangent at (x) of the curve C_u and the directrix of the first kind is

$$\left(\frac{\partial x}{\partial u}\right),$$

and the intersection of the tangent at (x) of the curve C_v and the canonical edge of the first kind is

$$\left(\frac{\partial x}{\partial v}\right).$$

The developpables of the congruence formed by the line which passes through the above two points correspond to a conjugate net.

Hence, we have the following theorem.

For a surface of the second kind which admits the projective deformation, the developables of the congruence formed by the lines which pass through the point where one of the asymptotic tangents at (x) meets the directrix of the first kind of (x), and through the point where the other of the asymptotic tangents at (x) meets the canonical edge of the first kind of (x), correspond to a conjugate net.

The inverse of this theorem is also true.

For then, we must have one of the equations

$$\frac{\partial^2}{\partial u \partial v} \log a = 0 \qquad \text{and} \qquad \frac{\partial^2}{\partial u \partial v} \log b = 0,$$

which we can reduce by the transformations of § 4 to

$$a = 1 \qquad \text{and} \qquad b = 1 \qquad \text{respectively.}$$

On the Projective Deformation of Curved Surfaces. II.

By

Jôyô Kanitani.

(Received April 19, 1922.)

Introduction.

In a preceding paper[1] we proved that for a curved surface (S) which is given by the system

$$\left.\begin{aligned} \frac{\partial^2 x}{\partial u^2} &= b\frac{\partial x}{\partial v} + f x, \\ \frac{\partial^2 x}{\partial v^2} &= a\frac{\partial x}{\partial u} + g x. \end{aligned}\right\} \qquad \text{(1)}$$

admits the projective deformation, it is necessary and sufficient that by the transformation

$$\bar{x} = C\sqrt{\alpha_u \beta_v}\, x, \qquad \bar{u} = \alpha(u), \qquad \bar{v} = \beta(v)$$

(where C is an arbitrary constant, $\alpha(u)$ and $\beta(v)$ are functions of u only and v only respectively, and the suffixes u and v denote the differentiation by u and v respectively) which does not alter the form of the equations (1), a and b are reduced so as to satisfy one of the relations

$$1^\circ \qquad \frac{\partial a}{\partial u} = \frac{\partial b}{\partial v}.$$

$$2^\circ \qquad a \text{ or } b = 1.$$

[1] Mem. Coll. Sci., Kyoto, **5**, 1–28 (1922).

In the former case, the surface (S) is of the first kind, while in latter case the surface is of the second kind.

Consider *a* congruence Γ conjugate to (S), that is, *a* congruence of straight lines such that

1° lines of the congruence are in the one to one correspondence with points on (S). Accordingly, to a surface which is generated by the lines of the congruence corresponds a curve on (S)

2° the lines of the congruence are in the tangent plane of (S) at the corresponding point

3° to a developpable surfaces of the congruence correspond a conjugate net on (S).

In this paper we shall prove, first, that if the said congruence is *a W*-congruence (that is, if asymptotic lines of two sheets of the focal surfaces correspond) and moreover, asymptotic lines of (S) and the focal surfaces also correspond, then the surface (S), all the Laplace-Darboux transforms of (S) with respect to the conjugate net in 3° and the focal surfaces of the congruence admit the projective deformation and the conjugate net in 3° is that of the projective deformation. Next, we shall prove that if there is an isothermal conjugate net on a surface (S) such that, the asymptotic lines of (S) and the minus-first or first Laplace-Darboux transform of (S) with respect to that conjugate net correspond, then (S) admits projective deformation and the conjugate net is that of projective deformation.

§ I.

Let us take as the parameter curves the conjugate net which corresponds to the developpable surfaces of the congruence. Then, as Green[1] has shown, the homogeneous cordinates (x) of a point (P) on (S) may be regarded as a fundamental system of solutions of completely integrable system of simultaneous partial differential equations of the form

$$\left.\begin{aligned} \frac{\partial^2 x}{\partial u^2} &= a\frac{\partial^2 x}{\partial v^2} + b\frac{\partial x}{\partial u} + c\frac{\partial x}{\partial v} + d\,x, \\ \frac{\partial^2 x}{\partial u \partial v} &= b'\frac{\partial x}{\partial u} + c'\frac{\partial x}{\partial v} + d'x, \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots (2)$$

[1] Amer. J. Math., **37**, 215–246, (1915).

From (2) we have

$$\frac{\partial^3 x}{\partial u^3} = \alpha^{(1)}\frac{\partial^2 x}{\partial v^2} + \beta^{(1)}\frac{\partial x}{\partial v} + \gamma^{(1)}\frac{\partial x}{\partial v} + \delta^{(1)}x,$$

$$\frac{\partial^3 x}{\partial u^2 \partial v} = \alpha^{(2)}\frac{\partial^2 x}{\partial v^2} + \beta^{(2)}\frac{\partial x}{\partial v} + \gamma^{(2)}\frac{\partial x}{\partial v} + \delta^{(2)}x,$$

$$\frac{\partial^3 x}{\partial u \partial v^2} = \alpha^{(3)}\frac{\partial^2 x}{\partial v^2} + \beta^{(3)}\frac{\partial x}{\partial u} + \gamma^{(3)}\frac{\partial x}{\partial v} + \delta^{(3)}x,$$

$$\frac{\partial^3 x}{\partial v^3} = \alpha^{(4)}\frac{\partial^2 x}{\partial v^2} + \beta^{(4)}\frac{\partial x}{\partial u} + \gamma^{(4)}\frac{\partial x}{\partial v} + \delta^{(4)}x.$$

where

$$\left.\begin{aligned}
&\alpha^{(1)} = a(b+c') + a_u, \\
&\beta^{(1)} = b_u + ab'_v + d + b^2 + b'(c+ab'), \\
&\gamma^{(1)} = c_u + ac'_v + ad' + bc + c'(c+ab'), \\
&\delta^{(1)} = d_u + ad'_v + bd + d'(c+ab'), \\
&\alpha^{(2)} = ab', \qquad \beta^{(2)} = b'(b+c') + b'_u + d', \\
&\gamma^{(2)} = b'c + c'^2 + c'_u, \qquad \delta^{(2)} = b'd + c'd' + d'_u, \\
&\alpha^{(3)} = c', \qquad \beta^{(3)} = b'^2 + b'_v, \\
&\gamma^{(3)} = b'c' + c'_v + d', \qquad \delta^{(3)} = b'd' + d'_v, \\
&\alpha^{(4)} = \frac{1}{a}\left(ab' - c - a_v\right), \qquad \beta^{(4)} = \frac{1}{a}\left(b'c' + b'_u - b_v + d'\right), \\
&\gamma^{(4)} = \frac{1}{a}\left[b'c + c'(c'-b) + c'_u - c_v - d\right], \\
&\delta^{(4)} = \frac{1}{a}\left[b'd + d'(c'-b) + d'_u - d_v\right].
\end{aligned}\right\} \quad (3)$$

The conditions of complete integrability are

$$\left.\begin{aligned}
&\gamma^{(3)} + \alpha^{(3)}_v = a\beta^{(4)} + \alpha^{(4)}_u, \\
&(\alpha^{(3)} - b)\beta^{(4)} + b'\beta^{(4)} + \beta^{(3)}_v = \alpha^{(4)}\beta^{(3)} + b'\gamma^{(4)} + \beta^{(4)}_u + \delta^{(4)}, \\
&c'\beta^{(3)} + \gamma^{(3)}_v + \delta^{(3)} = \alpha^{(4)}\gamma^{(3)} + c\gamma^{(4)} + \gamma^{(4)}_u, \\
&\alpha^{(3)}\delta^{(4)} + d'\beta^{(3)} + \delta^{(3)}_v = \alpha^{(4)}\delta^{(3)} + d\beta^{(4)} + d'\gamma^{(4)} + \delta^{(4)}_u.
\end{aligned}\right\} \quad (4)$$

In these formulas the suffixes u and v denote the differentiations by u and v respectively.

The first of (4), in virtue of equation (3), becomes

$$\frac{\partial}{\partial v}\left(b+2c'\right)=\frac{\partial}{\partial u}\left(2b'-\frac{c}{a}-\frac{\partial \log a}{\partial v}\right).$$

Hence we can find a function $p(u,v)$ by a quadrature such that

$$\left.\begin{aligned} b+2c' &= \frac{\partial p}{\partial n}, \\ 2b'-\frac{c}{a}-\frac{\partial \log a}{\partial v} &= \frac{\partial p}{\partial v}. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

By the transformation

$$x=\lambda \bar{x},$$

where λ is subjected to the conditions

$$\frac{1}{\lambda}\frac{\partial \lambda}{\partial u}=\frac{1}{4}\frac{\partial p}{\partial u}, \qquad \frac{1}{\lambda}\frac{\partial \lambda}{\partial v}=\frac{1}{4}\frac{\partial p}{\partial v},$$

the system (2) is transformed to a system of the same from with the confficients

$$\left.\begin{aligned} &A=a, \qquad B=b-\frac{1}{2}\frac{\partial p}{\partial u}, \qquad C=c+\frac{a}{2}\frac{\partial p}{\partial v}, \\ &D=d+\frac{1}{4}p\frac{\partial p}{\partial u}+\frac{1}{4}c\frac{\partial p}{\partial v}-\frac{1}{4}\frac{\partial^2 p}{\partial u^2}+\frac{1}{4}a\frac{\partial^2 p}{\partial v^2}-\frac{1}{16}\left(\frac{\partial p}{\partial u}\right)^2 \\ &\qquad\qquad +\frac{1}{16}a\left(\frac{\partial p}{\partial v}\right)^2, \\ &B'=b'-\frac{1}{4}\frac{\partial p}{\partial v}, \qquad C'=c'-\frac{1}{4}\frac{\partial p}{\partial u}, \\ &D'=d'+\frac{1}{4}b'\frac{\partial p}{\partial u}+\frac{1}{4}c'\frac{\partial p}{\partial v}-\frac{1}{4}\frac{\partial^2 p}{\partial u \partial v}-\frac{1}{16}\frac{\partial p}{\partial u}\frac{\partial p}{\partial v}. \end{aligned}\right\} \quad (7)$$

These coefficients are seminvariants of (2), and the new system is said to be in its canonical form. The relations

$$B+2C'=0, \qquad 2AB'-C-\frac{\partial A}{\partial v}=0,$$

which follow from (6), characteristic of this canonical form.

Any proper transformation of the form

$$\bar{u}=\varphi(u), \qquad \bar{v}=\psi(v). \quad \ldots\ldots\ldots\ldots\ldots (7)$$

affects only the parametric representation of the conjugate net given (2),

but leaves the net itself unchanged. The invariants of the net are those functions of the seminvariants, which remain unchanged by transformations of the form (7), except for a factor. The fundamental invariants are

$$a, \qquad \mathfrak{B}' = B' - \frac{3}{8}\frac{1}{a}\frac{\partial a}{\partial v}, \qquad \mathfrak{C}' = C' + \frac{1}{8}\frac{1}{a}\frac{\partial a}{\partial u},$$

$$\mathfrak{D}' = D' + B'C', \quad \mathfrak{D} = D - \left(B'\frac{\partial a}{\partial v} - a\frac{\partial B'}{\partial v}\right) - \frac{\partial C'}{\partial u} + 3\left(aB'^2 - C'^2\right).$$

These satisfy the equations

$$\bar{a} = \frac{\psi_v^2}{\varphi_u^2}, \qquad \overline{\mathfrak{B}'} = \frac{1}{\psi_v}\mathfrak{B}', \qquad \overline{\mathfrak{C}'} = \frac{1}{\varphi_u}\mathfrak{C}',$$

$$\overline{\mathfrak{D}} = \frac{1}{\varphi_u^2}\mathfrak{D}, \qquad \overline{\mathfrak{D}'} = \frac{1}{\varphi_u\psi_v}\mathfrak{D}'.$$

The second and third of the conditions (4) may be expressed in terms of the invariants as following.

$$\left.\begin{aligned}
&\frac{\partial \mathfrak{D}}{\partial v} - 2\frac{\partial \mathfrak{D}'}{\partial u} + \frac{\mathfrak{D}'}{a}\frac{\partial a}{\partial u} \\
&\quad = \frac{\partial}{\partial u}\left(\frac{\partial \mathfrak{B}'}{\partial u} + \frac{\partial \mathfrak{C}'}{\partial v} + \frac{1}{4}\frac{\partial^2 \log a}{\partial u \partial v}\right) - 4\mathfrak{C}'\left(\frac{\partial \mathfrak{B}'}{\partial u} + 3\frac{\partial \mathfrak{C}'}{\partial v}\right) \\
&\qquad - \frac{1}{2a}\frac{\partial a}{\partial u}\left(\frac{\partial \mathfrak{B}'}{\partial u} + \frac{\partial \mathfrak{C}'}{\partial v} + \frac{1}{4}\frac{\partial^2 \log a}{\partial u \partial v}\right), \\
&\frac{\partial}{\partial u}\left(\frac{\mathfrak{D}}{a}\right) + 2\frac{\partial \mathfrak{D}'}{\partial v} + \frac{\mathfrak{D}'}{a}\frac{\partial a}{\partial v} \\
&\quad = -\frac{\partial}{\partial v}\left(\frac{\partial \mathfrak{B}'}{\partial u} + \frac{\partial \mathfrak{C}'}{\partial v} - \frac{3}{4}\frac{\partial^2 \log a}{\partial u \partial v}\right) + 4\mathfrak{B}'\left(3\frac{\partial \mathfrak{B}'}{\partial u} + \frac{\partial \mathfrak{C}'}{\partial v}\right) \\
&\qquad - \frac{1}{2a}\frac{\partial a}{\partial u}\left(\frac{\partial \mathfrak{B}'}{\partial u} + \frac{\partial \mathfrak{C}'}{\partial v} - \frac{3}{4}\frac{\partial^2 \log a}{\partial u \partial v}\right).
\end{aligned}\right\} \quad (8)$$

§ 2.

Consider two points ρ and σ such that

$$\left.\begin{aligned}
\rho &= \theta\frac{\partial x}{\partial u} - \frac{\partial \theta}{\partial u}x, \\
\sigma &= \theta\frac{\partial x}{\partial v} - \frac{\partial \theta}{\partial v}x,
\end{aligned}\right\} \quad (9)$$

where θ is a solution of the partial differential equation

$$\frac{\partial^2\theta}{\partial u\partial v} = b'\frac{\partial\theta}{\partial u} + c'\frac{\partial\theta}{\partial v} + d'\theta. \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (10)$$

Then we have seen that[1] congruences whose developpables correspond to the curves C_u (v-const.) and C_v (u-const.) are those which are formed by such lines as $\rho\sigma$ and the surfaces which are described by ρ and σ are focal surfaces.

Put

$$w = a\frac{\partial^2\theta}{\partial v^2} + b\frac{\partial\theta}{\partial u} + d\theta - \frac{\partial^2\theta}{\partial u^2},$$

$$\phi = a^{(1)}\frac{\partial^2\theta}{\partial v^2} + \beta^{(1)}\frac{\partial\theta}{\partial u} + \gamma^{(1)}\frac{\partial\theta}{\partial v} + \delta^{(1)}\theta - \frac{\partial^3\theta}{\partial u^3},$$

$$\psi = a^{(4)}\frac{\partial^2\theta}{\partial v^2} + \beta^{(4)}\frac{\partial\theta}{\partial u} + \gamma^{(4)}\frac{\partial\theta}{\partial\theta} + \delta^{(4)}\theta - \frac{\partial^3\theta}{\partial v^3},$$

Then we have from (3), (4) and (10)

$$\frac{\partial^3\theta}{\partial u^2\partial v} = -b'w + a^{(2)}\frac{\partial^2\theta}{\partial v^2} + \beta^{(2)}\frac{\partial\theta}{\partial u} + \gamma^{(2)}\frac{\partial\theta}{\partial v} + \delta^{(2)}\theta,$$

$$\frac{\partial^3\theta}{\partial u\partial v^2} = a^{(3)}\frac{\partial^2\theta}{\partial v^2} + \beta^{(3)}\frac{\partial\theta}{\partial u} + \gamma^{(3)}\frac{\partial\theta}{\partial v} + \delta^{(3)}\theta,$$

$$\frac{\partial w}{\partial u} = -bw + \phi,$$

$$\frac{\partial w}{\partial v} = b'w - a\psi,$$

$$\frac{\partial\psi}{\partial u} = c'\psi - \beta^{(4)}w,$$

$$\frac{\partial^2 w}{\partial u\partial v} = b'\frac{\partial w}{\partial u} + \left(c' + \frac{a_u}{a}\right)\frac{\partial w}{\partial v} + \left(2b'_u - b_v + d' - \frac{b'a_u}{a}\right)w. \quad \cdots\cdots\cdots\cdots (11)$$

From (2), (9) and (10) we have

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[1] J. Kanitani, Mem. Coll. Sci., Kyoto, **5**, 329, (1922).

$$\left.\begin{aligned}\frac{\partial\rho}{\partial v}&=\left(b'+\frac{\theta_v}{\theta}\right)\rho+\left(c'-\frac{\theta_v}{\theta}\right)\sigma,\\ \frac{\partial\sigma}{\partial u}&=\left(b'-\frac{\theta_v}{\theta}\right)\rho+\left(c'+\frac{\theta_u}{\theta}\right)\sigma,\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (12)$$

from which follow

$$\left.\begin{aligned}\frac{\partial^2\rho}{\partial u\partial v}&=\left(b'+\frac{\theta_v}{\theta}\right)\frac{\partial\rho}{\partial u}+\left(c'+\frac{\theta_u}{\theta}+\frac{\tau_u}{\tau}\right)\frac{\partial\rho}{\partial v}\\ &\quad+\left[b'_u+b'c'+d'-\left(b'+\frac{\theta_v}{\theta}\right)\left(c'+\frac{\theta_u}{\theta}+\frac{\tau_u}{\tau}\right)\right]\rho,\\ \frac{\partial^2\sigma}{\partial u\partial v}&=\left(b'+\frac{\theta_v}{\theta}+\frac{\tau'_v}{\tau'}\right)\frac{\partial\sigma}{\partial u}+\left(c'+\frac{\theta_u}{\theta}\right)\sigma_v\\ &\quad+\left[c'_v+b'c'+d'-\left(b'+\frac{\theta_v}{\theta}+\frac{\tau'_v}{\tau'}\right)\left(c'+\frac{\theta_u}{\theta}\right)\right]\rho,\end{aligned}\right\}\quad(13)$$

where

$$\tau\equiv c'-\frac{\theta_u}{\theta},\qquad\qquad\tau'\equiv b'-\frac{\theta_v}{\theta}.$$

Now we shall prove that ρ and σ satisfy the equations of the form

$$\frac{\partial^2\rho}{\partial u\partial v}=A_1\frac{\partial\rho}{\partial v^2}+B_1\frac{\partial\rho}{\partial u}+C_1\frac{\partial\rho}{\partial v}D_1\rho\quad\ldots\ldots\ldots\ldots\ldots\ldots\ (14)$$

and

$$\frac{\partial^2\sigma}{\partial u\partial v}=A_2\frac{\partial\sigma}{\partial v^2}+B_2\frac{\partial\sigma}{\partial u}+C_2\frac{\partial\sigma}{\partial v}+D_2\sigma\quad\ldots\ldots\ldots\ldots\ldots\ldots\ (15)$$

respectively,

where A_1, B_1, ...; A_2, B_2, ... are functions of a, b, ..., θ and their derivatives.

In fact, we have form (2), (9) and (10)

$$\rho=\theta\frac{\partial x}{\partial u}-\frac{\partial\theta}{\partial u}x,$$

$$\frac{\partial\rho}{\partial u}=\theta\frac{\partial^2x}{\partial u^2}-\frac{\partial^2\theta}{\partial u^2}x=a\frac{\partial\sigma}{\partial v}+b\rho+c\sigma+wx,$$

$$\frac{\partial\rho}{\partial v}=\left(b'+\frac{\theta_v}{\theta}\right)\rho+\left(c'-\frac{\theta_v}{\theta}\right)\sigma,$$

$$\frac{\partial^2\rho}{\partial u^2}=\frac{\theta_u}{\theta}\frac{\partial\rho}{\partial u}+a^{(1)}\frac{\partial\rho}{\partial v}+\left(\beta^{(1)}-\frac{\theta_{uu}}{\theta}\right)\rho+\gamma^{(1)}\sigma+\phi x,$$

$$\frac{\partial^2\rho}{\partial v^2}=\left(b'+\frac{\theta_v}{\theta}\right)\frac{\partial\rho}{\partial v}+\left(c^1-\frac{\theta_u}{\theta}\right)\frac{\partial\sigma}{\partial v}+\frac{\partial}{\partial v}\left(b^1+\frac{\theta_v}{\theta}\right)\rho$$

$$+\frac{\partial}{\partial v}\left(c^1-\frac{\theta_u}{\theta}\right)\sigma,$$

from which follows

$$\frac{\partial^2\rho}{\partial u^2}+\frac{a\left\{w_u-\left(c'+\frac{a_u}{a}\right)\right\}}{c'-\frac{\theta_v}{\theta}}\frac{\partial^2\rho}{\partial v^2}$$

$$=\left(b+\frac{\theta_u}{\theta}+\frac{w_u}{w}\right)\frac{\partial\rho}{\partial u}+\frac{a\left\{w_u-\left(c'+\frac{a_u}{a}\right)\right\}\left(b'+\frac{\theta_v}{\theta}\right)}{c'-\frac{\theta_u}{\theta}}$$

$$+\left[\frac{\beta^{(3)}-\frac{\partial_{uu}}{\theta}+a\left\{w_u-\left(c'+\frac{a_u}{a}\right)\right\}}{c'-\frac{\theta_u}{\theta}}\frac{\partial}{\partial v}\left(b'+\frac{\theta_v}{\theta}\right)-b\left(b+\frac{w_u}{w}\right)\right]\rho$$

$$+\left[\gamma^{(1)}\frac{a\left\{w_u-\left(c'+\frac{a_u}{a}\right)\right\}}{c'-\frac{\theta_u}{\theta}}\frac{\partial}{\partial v}\left(c'-\frac{\theta_u}{\theta}\right)-c\left(b+\frac{w_u}{w}\right)\right]\sigma. \quad \ldots\ldots (16)$$

Substituting

$$\sigma=\frac{\frac{\partial\rho}{\partial v}}{c'-\frac{\theta_u}{\theta}}-\frac{b'+\frac{\theta_v}{\theta}}{c'+\frac{\theta_u}{\theta}}\rho$$

in (17), we have an equation of the form (14), and

$$A_1=\frac{a\left(c'+\frac{\theta_u}{\theta}-\frac{w_u}{w}\right)}{c'-\frac{\theta_u}{\theta}}. \quad \ldots\ldots\ldots\ldots (17)$$

Similarly we can prove for σ and we have

$$A_2=\frac{a\left(b'-\frac{\theta_v}{\theta}\right)}{b'-\frac{w_v}{w}}. \quad \ldots\ldots\ldots\ldots (17')$$

Asymptotic lines of (S) and the surfaces which are described by ρ and σ are given by

$$a\,du^2 + dv^2 = 0,$$
$$A_1\,du^2 + dv^2 = 0,$$
$$A_2\,du^2 + dv^2 = 0. \qquad \text{respectively.}$$

If those asymptotic lines correspond, we must have

$$a = A_1 = A_2,$$

or

$$a = \frac{a\left(c' + \frac{a_u}{a} - \frac{w_u}{w}\right)}{c' - \frac{\theta_u}{\theta}} = \frac{b' + \frac{\theta_v}{\theta}}{b' - \frac{w_v}{w}} \quad \cdots\cdots\cdots\cdots\cdots (18)$$

from which follow

$$w = f(u)\theta,$$
$$a = \frac{f(u)}{g(v)}, \quad \cdots\cdots\cdots\cdots\cdots (19)$$

where $f(u)$ and $g(v)$ are functions of u only and v only respectively.

Substituting the above values of w in the equation (11), we have

$$\frac{\partial^2\theta}{\partial u\partial v} = b'\frac{\partial\theta}{\partial u} + c'\frac{\partial\theta}{\partial v} + (2b'_u - b_v + d')\theta = 0.$$

From this equation and (10) we have

$$\theta(2b'_u - b_v) = 0.$$

If $\theta = 0$, the points (ρ) and (σ) coincide with (x).

Hence we must have

$$2\frac{\partial b'}{\partial u} - \frac{\partial b}{\partial v} = 0. \quad \cdots\cdots\cdots\cdots\cdots (20)$$

The conditions (19) and (20) may be expressed in terms of invariants as following.

$$\left.\begin{aligned} \frac{\partial^2 \log a}{\partial u\partial v} &= 0, \\ \frac{\partial \mathfrak{B}'}{\partial u} + \frac{\partial \mathfrak{E}'}{\partial v} &= 0. \end{aligned}\right\} \cdots\cdots\cdots\cdots\cdots (21)$$

Reciprocally, if

$$a = \frac{f(u)}{g(v)}$$

and

$$2\frac{\partial b'}{\partial u}-\frac{\partial b}{\partial v}=0,$$

the system of equations

$$\left.\begin{aligned}\frac{\partial^2\theta}{\partial u^2}&=a\frac{\partial^2\theta}{\partial v^2}+b\frac{\partial^2\theta}{\partial u}+c\frac{\partial\theta}{\partial v}+\{d-f(u)\}\theta,\\ \frac{\partial^2\theta}{\partial u\partial v}&=b'\frac{\partial\theta}{\partial\theta}+c'\frac{\partial\theta}{\partial v}+d'\theta\end{aligned}\right\}\quad\cdots\cdots\cdots(22)$$

is completely integrable by (4).

Therefore, if we take a particular solution of the system (22) and consider two points

$$\rho=\theta\frac{\partial x}{\partial u}-\frac{\partial\theta}{\partial u}x,$$

$$\sigma=\theta\frac{\partial x}{\partial v}-\frac{\partial\theta}{\partial u}x,$$

then the congruence formed by the lines $\rho\sigma$ satisfies our conditions.

§ 3.

By the transformation

$$\bar{u}=\int\sqrt{f(u)}\,du,\qquad \bar{v}=i\int\sqrt{g(v)}\,dv,$$

(21) become

$$\bar{a}=-1,$$

$$\frac{\partial\overline{\mathfrak{B}}'}{\partial\bar{u}}+\frac{\partial\overline{\mathfrak{C}}'}{\partial\bar{v}}=0.$$

Let us suppose that we have performed the above transformation. Then from (18) we have

$$w=k\theta,$$

where k is a constant.

Next, reduce the thus transformed system into its canonical form and denote by A, B, etc., the coefficients of that canonical form, then we have

$$\left.\begin{aligned} A &= -1, \\ B + 2C' &= 0, \\ 2B' + C &= 0, \\ \frac{\partial B'}{\partial u} + \frac{\partial C'}{\partial v} &= 0. \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (23)$$

Asymptotic lines of (S) are given by

$$du^2 - dv^2 = 0.$$

Hence, if we carry out the transformation

$$\xi = u - v, \qquad \mu = u + v,$$

the new parameter curves are asymptotic lines and the system (2) is transformed into

$$\frac{\partial^2 x}{\partial \xi^2} = \beta \frac{\partial x}{\partial \mu} + \gamma x,$$

$$\frac{\partial^2 x}{\partial \mu^2} = \alpha \frac{\partial x}{\partial \xi} + \gamma' x,$$

where

$$\alpha = (B' - C'), \qquad \beta = -(B' + C'),$$

$$\gamma = \frac{D + 2D'}{4}, \qquad \gamma' = \frac{D - 2D'}{4}.$$

Moreover, we have

$$\frac{\partial B'}{\partial u} = \frac{\partial B'}{\partial \xi} + \frac{\partial B'}{\partial \mu},$$

$$\frac{\partial C'}{\partial v} = -\frac{\partial C'}{\partial \xi} + \frac{\partial C'}{\partial \mu},$$

from which follows

$$\frac{\partial}{\partial \xi}\left(B' - C'\right) + \frac{\partial}{\partial \mu}\left(B' + C'\right),$$

which is equivalent to

$$\frac{\partial \alpha}{\partial \xi} = \frac{\partial \beta}{\partial \mu}.$$

Therefore we know that (S) admits the projective deformation and is of the first kind, and the curves $u = \text{const.}$ and $v = \text{const.}$ form the conjugate net of the projective deformation.

From (13) and (17), we know that when (S) is given by the said canonical form the surface which is described by (ρ) is given by the system

$$\left.\begin{aligned}
\frac{\partial^2\rho}{\partial u^2}+\frac{\partial^2\rho}{\partial v^2}&=2\left(\frac{\theta_u}{\theta}-C'\right)\frac{\partial\rho}{\partial u}+2\left(\frac{\theta_v}{\theta}+\frac{\tau_v}{\tau}-B'\right)\frac{\partial\rho}{\partial v}+(\bigstar)\rho,\\
\frac{\partial^2\rho}{\partial u\partial v}&=\left(B'+\frac{\theta_v}{\theta}\right)\frac{\partial\rho}{\partial u}+\left(C'+\frac{\theta_u}{\theta}+\frac{\tau_u}{\tau}\right)\frac{\partial\rho}{\partial v}\\
&\quad+\left[K-\left(C'+\frac{\theta_u}{\theta}+\frac{\tau_u}{\tau}\right)\left(B'+\frac{\theta_v}{\theta}\right)\right]\rho,
\end{aligned}\right\}\tag{23}$$

where

$$K=B'C'+D'-\frac{\partial C'}{\partial v},$$

$$\tau=C'-\frac{\theta_u}{\theta}.$$

Similary we know that the surface which is described by σ is given by the system

$$\left.\begin{aligned}
\frac{\partial^2\sigma}{\partial u^2}+\frac{\partial^2\sigma}{\partial v^2}&=2\left(\frac{\theta_u}{\theta}+\frac{\tau'_u}{\tau'}-C'\right)\frac{\partial\sigma}{\partial u}+2\left(\frac{\theta_v}{\theta}-B'\right)\frac{\partial\sigma}{\partial v}+(\bigstar)\sigma,\\
\frac{\partial^2\sigma}{\partial u\partial v}&=\left(B^1+\frac{\theta_v}{\theta}+\frac{\tau'_v}{\tau'}\right)\frac{\partial\sigma}{\partial u}+\left(C'+\frac{\theta_u}{\theta}\right)\frac{\partial\sigma}{\partial v}\\
&\quad+\left[H-\left(B'+\frac{\theta_v}{\theta}+\frac{\tau'_v}{\tau'}\right)\left(C'+\frac{\theta_v}{\theta}\right)\right]\sigma,
\end{aligned}\right\}\tag{24}$$

where

$$H=B'C'+D'-\frac{\partial B'}{\partial u},$$

$$\tau=B'-\frac{\theta_v}{\theta}.$$

From (24) and (25) we can easily see that the focal surfaces also admit the projective deformation.

§ 4.

Let (y) be the point on the minus-first Laplace-Darboux transform (S_{-1}) of (S) corresponding to (x).

Then we have from (2)

$$
\left.\begin{aligned}
y &= \frac{\partial x}{\partial u} - c'x,\\
\frac{\partial y}{\partial u} &= a\frac{\partial^2 x}{\partial v^2} + \left(b - c'\right)\frac{\partial x}{\partial u} + c\frac{\partial x}{\partial v} + (d - c'_u)x,\\
\frac{\partial y}{\partial v} &= \qquad\qquad b'\frac{\partial x}{\partial u} \qquad\qquad + (d' - c'_v)x,\\
\frac{\partial^2 y}{\partial u^2} &= \left(a^{(1)} - ac'\frac{\partial^2 x}{\partial v^2}\right) + \left(\beta^{(1)} - bc' - 2c'_u\right)\frac{\partial x}{\partial u} + \left(a^{(1)} - cc'\right)\frac{\partial x}{\partial v}\\
&\qquad\qquad + \left(\delta^{(1)} - c'd - c'_{uu}\right)x,\\
\frac{\partial^2 y}{\partial v^2} &= \left(b'^2 + b'_v\right)\frac{\partial x}{\partial u} + \left(b'c' + d' - c'_v\right)\frac{\partial x}{\partial v} + \left(d'_v + b'd' - c'_{vv}\right)x,
\end{aligned}\right\} \quad (26)
$$

from which follows

$$
\left.\begin{aligned}
&\frac{\partial^2 y}{\partial u^2} - A\frac{\partial^2 y}{\partial v^2} - \left(b + \frac{a_u}{a}\right)\frac{\partial y}{\partial u}\\
&\quad = \left[\beta^{(1)} - bc' - 2c'_u - A\left(b'^2 + b'_v\right) - \left(b - c'\right)\right]\frac{\partial x}{\partial u}\\
&\quad + \left[\delta^{(1)} - c'd - c'_{uu} - A\left(d'_v + b'd' - c'_{vv}\right)\left(d - c'_u\right)\left(b + \frac{a_u}{a}\right)\right]x,
\end{aligned}\right\}
$$

where

$$
A = \frac{a\left\{\dfrac{\partial}{\partial u}\left(\dfrac{c}{a}\right) + 2\dfrac{\partial c'}{\partial v} + K\right\}}{K}.
$$

Eliminating x, $\dfrac{\partial x}{\partial u}$ from the first, the third and last equations, we have the equation of the form

$$
\frac{\partial^2 y}{\partial u^2} = A\frac{\partial^2 y}{\partial v^2} + B\frac{\partial y}{\partial v^2} + C\frac{\partial y}{\partial v} + Dy,
$$

where

$$
A = \frac{a\left\{\dfrac{\partial}{\partial u}\left(\dfrac{c}{a}\right) + 2\dfrac{\partial c'}{\partial v} + K\right\}}{K}.
$$

And from the equations

$$
y = \frac{\partial x}{\partial u} - c'x,
$$

$$\frac{\partial^2 x}{\partial u \partial v} = b' \frac{\partial x}{\partial v} + c' \frac{\partial x}{\partial v} + d' x.$$

we have

$$\frac{\partial^2 y}{\partial u \partial v} = b' \frac{\partial y}{\partial u} + \left(c' + \frac{K_u}{K} \right) \frac{\partial y}{\partial v} + \left[b'_u + K - b' \left(c' + \frac{K_u}{K} \right) \right] y. \quad \ldots\ldots (27)$$

When the asymptotic lines of (S) and (S_1) correspond, we must have

$$a = A,$$

or

$$\frac{\partial}{\partial u}\left(\frac{c}{a}\right) - \frac{\partial c'}{\partial v} = 0,$$

which is equivalent to

$$2\frac{\partial b'}{\partial u} - \frac{\partial b}{\partial v} - \frac{\partial^2}{\partial u \partial v} \log a = 0. \quad \ldots\ldots (28)$$

If the curves C_u and C_v form an isothermal conjugate net, we must have[1]

$$\frac{\partial^2}{\partial u \partial v} \log a = 0. \quad \ldots\ldots (29)$$

Therefore, from (28) and (29) we know that *if there is an isothermal conjugate net on a surface* (S) *such that, the asymptotic lines of* (S) *and the minus-first Laplace-Darboux transform of it with respect to that conjugate net correspond,* (S) *admits the projective deformation and the conjugate net is that of projective deformation.*

This theorem also hold, if we take the first instead of the minus-first Laplace-Darboux transform of (S).

When (S) admits the projective deformation and given by the system which is its canonical form explained in § 3, from (8), (23) and (26) we can see that the minus-first Laplace-Darboux transform of (S) is given by the system

$$\left.\begin{aligned} \frac{\partial^2 y}{\partial u^2} + \frac{\partial^2 y}{\partial v^2} &= -2C' \frac{\partial y}{\partial u} + 2\left(\frac{K_v}{K} - B' \right) \frac{\partial y}{\partial v} + (\bigstar) y, \\ \frac{\partial^2 y}{\partial u \partial v} &= B' \frac{\partial y}{\partial u} + \left(C' + \frac{K_u}{K} \right) \frac{\partial y}{\partial v} + \left[B'_u + K - B' \left(C' + \frac{K_u}{K} \right) \right] y. \end{aligned}\right\} \quad (30)$$

Similarly we can see that the first transform is given by the system

[1] Wilcynski, Amer. J. Math., **42**, 211, (1920).

$$\left.\begin{aligned}\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}&=2\left(\frac{H_u}{H}-C'\right)\frac{\partial z}{\partial u}-2B'\frac{\partial z}{\partial v}+(\bigstar)z,\\ \frac{\partial^2 z}{\partial u\partial v}&=\left(B'+\frac{H_v}{H}\right)\frac{\partial z}{\partial v}+C'\frac{\partial z}{\partial v}+\left[C_v'+H-C'\left(B'+\frac{H_v}{H}\right)\right]z.\end{aligned}\right\}\quad(31)$$

From (30) and (31) we know that the minus-first and the first Laplace-Darboux transforms with respect to the conjugate net of the projective deformation admit the projective deformation and, moreover, the conjugate net of the projective deformation is transformed into the conjugate net of the projective deformation, accordingly the same is true for all the Laplace-Darboux transforms with respect to the conjugate net of the projective deformation.

On the Synthesis of the Phosphoric Acid Esters, I.

By

Shigeru Komatsu and Shinichiro Kumamoto.

(Received May 12, 1922.)

Phosphoric acid esters are interesting substances in the chemical study of alcoholic fermentation and of living cells, specially from the biochemical standpoint.[1] As to the synthesis of phosphoric acid esters with hydroaromatic alcohols, only that of the innosite was described by S. Posternak[2] and R. J. Anderson.[3] The esters of other mono- and di- atomic alcohols of polymethylene series, however, have not yet been described. In our present article we have undertaken the preparation of the hexahydrophenol esters.

Phosphoric acid hexahydrophenol ester was prepared, in the same manner with innosite ester studied by S. Posternak,[2] from phosphoric acid and hexahydrophenol, phosphorpentoxide using as a catalyser; to a cold and uniform mixture of 60 grms phosphoric acid and 84 grms phosphorpentoxide were added 60 grms hexahydrophenol prepared from phenol by the catalytic hydrogenation following the Prof. P. Sabatier's directions.[4] The mixture was left to stand at an ordinary temperature

1 A. Harden, Alcoholic fermentation (1911), 38.
H. Euler, P. Lindner, Chemie der Hefe u. den alkolischen Gärung (1915), 158.
F. Czapek, Biochemie der Pflanzen I, 764; III, 483 (1921).
W. Glikin, Chemie der Fette Lipoide u. Wachsarten, I, 324 (1912).
H. Maclean, Lecithin and allied substances (1918) 170.
E. B. Forbes and M. H. Keith, A review of the literature of phosphorus compounds in animal metabolism (1914). R. Rose, A resumé of the literture on inositephosphoric acid & c. Biochem J. **2**.21 (1912).

2 C. R. **168**, 1216; **169**, 79 (1919); Hel. chim. Acta, **4**, 150 (1921).

3 J. Biol. Ch., **43**, 167 (1920).

4 C. R. **137**, 1025 (1903).

for a few days, and then water was added, whereupon a brown oil separated. The acid solution separated from the upper oily layer, was neutralized with barium hydroxide solution, and the excess of baryta remaining in the solution was removed by passing carbon dioxide into the solution. The filtrate separated from the insoluble barium salt, was evaporated to dryness on a water bath, and the residue was recrystallized from the water solution and a white crystalline substance was obtained which was confirmed by elementary analysis to be the barium salt of mono-hexahydrophenol phosphoric acid ester.

0.1218 grms. of the substance gave 0·1011 grms. CO_2 & 0·0419 grms. H_2O
0·2030 grms. of the substance gave 0·1492 grms. $BaSO_4$
0·1366 grms. of the substance gave 0·0464 grms. $Mg_2P_2O_7$

	Calc. for $C_6H_{11}O\diagdown\ Ba\diagup O\!-\!P\!=\!O$	Found.
C	22·83	22·64
H	3·49	3·82
Ba	43·56	43·25
PO_4	30·12	29·05

The oily part separated from the aqueous layer was dissolved entirely into ether, and the ethereal solution was then treated with barium hydroxide to remove phosphoric acid ester as the barium salt from hexahydrophenol unchanged. An excess of barium hydroxide in the aqueous extract was removed by means of carbon dioxide, and the water solution was evaporated to dryness on a water bath. The white residue thus obtained was extracted with anhydrous ether in Soxley's apparatus and the insoluble residue was confirmed to be the barium salt of the mono-ester by analysis.

0·1316 grms. of the substance gave 0.1106 grms. CO_2, 0·0441 grms. H_2O and 0·0950 grms. ash.

	Calc. for $C_6H_{11}O\diagdown\ Ba\diagup O\!-\!P\!=\!O$	Found.
C	22·83	22·92
H	3·49	3·72
Ash	73·68	72·19

The ether extract was divided into two parts: to the one, dry ammonia gas was passed and a white precipitate was obtained which

after being converted into silver salt with silver nitrate was analysed, with the following results.

0·1205 grms. of the substance gave 0·0830 grms. CO_2 and 0·0306 grms. H_2O.

	Calc. for $\begin{matrix} C_6H_{11}O \\ AgO \\ AgO \end{matrix}\!\!>\!P=O$	Found.
C	18·28	18·79
H	2·79	2·82

The other part of the ethereal extract was evaporated carefully on a water bath almost to dryness, and allowed to stand for three weeks in a calcium chloride-paraffin desiccator. Thus a free monoester was obtained, consisting of colorless crystals melting at 77–78°.

0·1094 grms. of the substance gave 0·1555 grms. CO_2 and 0·0724 grms. H_2O.

	Calc. for $\begin{matrix} C_6H_{11}O \\ HO \\ HO \end{matrix}\!\!>\!P=O$	Found.
C	40·00	39·01
H	7·22	7·35

The free ester was soluble in water, alcohol and ether, and its barium and ammonium salts were soluble easily in water, but its silver salt was insoluble in water. When heated, the free ester and also the barium salt were observed to decompose into phosphoric acid and tetrahydrobenzene.

In order to prepare the tri- and di-esters, 50 grms. hexahydrophenol and 35 grms. phosphoroxychloride were mixed and the mixture was left to stand at an ordinary temperature for a few weeks. When the generation of hydrogenchloride gas had ceased, the reaction product was treated with water, caustic soda solution successively and the insoluble oily residue was subjected to steam distillation. The residue in a flask was distilled and collected the distillate boiling at 230–280°. The distillate was again fractionated under 6–7 mm. pressure and the following two fractions were obtained:

I. First fract. Bp. 100–125°; yield, 2·5 grms.
II. Second fract. Bp. 126–127°; yield, 2·5 grms.

The first fraction was a colorless and the second was a light-yellowish-orange liquid and possessed a perfume like an orange. Density and chemical composition of both fractions were determined, and the following results were obtained.

I. $d_4^{25}=0.9903$; $n_D^{25}=1.5028$.

1. 0·1090 grms. of subst. gave 0·3092 grms. CO_2 & 0·0986 grms. H_2O.
2. 0·1220 grms. „ 0·3472 „ 0·1087 „
3. 0·1196 grms. „ 0·3402 „ 0·1065 „

	1	2	3	Mean
C	77·36	77·62	77·58	77·52
H	10·05	9·90	9·89	9·95

II. $d_4^{25}=0.9938$; $n_D^{25}=1.5026$.

1. 0·1050 grms. of subst. gave 0·3026 grms. CO_2 & 0.0862 grms. H_2O.
2. 0·1444 grms. „ 0·4168 „ 0·1176 „
3. 0·1260 grms. „ 0·3646 „ 0·1081 „

	1	2	3	Mean
C	78·60	78·72	78·92	78·75
H	9·12	9·05	9·53	9·23

The chemical and physical properties of the second fraction agree with those of the di-hexahydrophenyl ether requiring C=79·12; H=12·09, which was obtained by W. Ipatiew and O. Philipow[1] from phenyl ether by catalytic hydrogenation in the presence of nickel oxide under high pressure.

It is a note-worthy fact to mention here that aromatic alcohol such as phenol easily yields with phosphoric acid, mono-, di- and tri-esters[2] and hexahydrophenol, on the contrary, only one mono-ester.

March 1922. Biochemical Laboratory, Institute of Chemistry.

1 Ber. D. Chem. Gesell., **41**, 1001 (1908).

2 Williamson and Serugam, Lieb. Ann., **92**, 31 (1854).
R. Hein, Ber. D. Chem. Gesell., **16**, 1763 (1883).

On the System: Sodium Sulphate—Sodium Chromate—Water. II.

By

Yukichi Osaka and Ryohei Yoshida.

(Received June 5, 1922.)

In our laboratory, I. Takeuchi[1] studied, on the suggestion of one of the authors, the heterogeneous equilibria in the system of sodium sulphate, sodium chromate and water at 15° and 25°. This article is a continuation of that work.

The transition points of the hydrates of sodium sulphate and chromate are as follows, as given in the literature:

$$Na_2SO_4 \cdot 10H_2O \underset{32.383^\circ}{\rightleftarrows} Na_2SO_4 + 10H_2O,^2$$

$$Na_2CrO_4 \cdot 10H_2O \underset{19.52^\circ}{\rightleftarrows} Na_2CrO_4 \cdot 6H_2O + 4H_2O,^3$$

$$Na_2CrO_4 \cdot 6H_2O \underset{26.6^\circ}{\rightleftarrows} Na_2CrO_4 \cdot 4H_2O + 2H_2O,^3$$

$$Na_2CrO_4 \cdot 4H_2O \underset{62.8^\circ}{\rightleftarrows} Na_2CrO_4 + 4H_2O.^3$$

Takeuchi has shown that the two salts have complete miscibilty as a solid solution at 15°, at which they are both decahydrate and isomorphous, but that at 25°, at which the sulphate is still decahydrate, while the chromate hexahydrate, the sulphate dissolves the chromate as a decahydrated solid solution to the extent of the molar fraction of 0·34, but the chromate hexahydrate can not take up the sulphate at all. He showed also that when the amount of the chromate in the solution exceeds a 77 molar percentage of the total salts, the sulphate separates

1 Mem. Col. Sci. Kyoto Imp. Univ., **1**, 249 (1915).

2 Richards and Wells, Zeitschr. physik. Chem., **43**, 471 (1903).

3 Richards and Kelly, J. Amer. Chem. Soc., **33**, 847 (1911).

out as the anhydrous salt. The present authors took up the further study of the equilibria in the same system at 28°, 31° and 33°.

Sodium sulphate, Jap. Pharm., was thrice recrystallised and used in fine crystals of the decahydrate. As to the chromate, the pure preparation of C. Merck was recrystallised and it was also prepared from the bichromate after Richards.[1] The limits of the temperature of the existence of the hexahydrate are narrow, and as it was found difficult to obtain it from its supersaturated solution when there was no sample of the hexahydrate to be used as the crystallisation nucleus, the following method was devised: The decahydrate is heated above 21° and some water is driven off from the fused mass by evaporation. It is then cooled below 19° and the solidified mass is heated again above 21°. The decahydrate fuses and the hexahydrate remains as crystals.

The experimental methods were the same as those of Takeuchi in the essential points.

The results of the experiments are given in Tables 1, 2, and 3 and Figures 1, 2, and 3, where the compositions of the solutions and residues (in some cases somewhat moist with the mother liquor) are represented by the formula:

$$100m\mathrm{H_2O}\ x\mathrm{Na_2CrO_4}\ (100-x)\ \mathrm{Na_2SO_4}.$$

TABLE I.

Temperature: 28·0°.

No.	Solution.		Residue.	
	x	m	x	m
1	0·00	22·10	0·00	10·00
2	8·15	21·47	0·84	10·58
3	25·15	20·32	4·96	9·98
4	26·18	19·66	4·94	10·60
5	33·39	18·29	5·48	9·81
6	40·51	17·37	—	—
7	44·09	16·87	14·35	10·14
8	47·87	16·51	12·17	9·80
9	54·88	14·68	12·32	7·79
10	55·55	14·76	12·76	7·78
11	54·70	14·64	10·63	7·00
12	55·91	14·88	6·69	1·88
13	56·40	15·03	8·05	2·23

[1] Richards and Kelly, *Loc. cit.*

No.	Solution.		Residue.	
	x	m	x	m
14	56·72	15·37	9·41	2·33
15	60·89	14·96	6·12	1·56
16	62·35	14·47	9·36	2·22
17	63·03	14·53	5·89	1·09
18	64·31	14·32	10·36	2·39
19	69·98	14·12	8·53	1·66
20	79·38	13·35	11·60	1·85
21	91·35	11·34	7·71	0·84
22	95·64	10·37	63·22	2·91
23	94·78	10·52	87·63	4·28
24	94·36	10·47	85·14	4·09
25	98·19	10·20	100·00	4·39
26	100·00	10·51	100·00	4·32

Fig. 1.

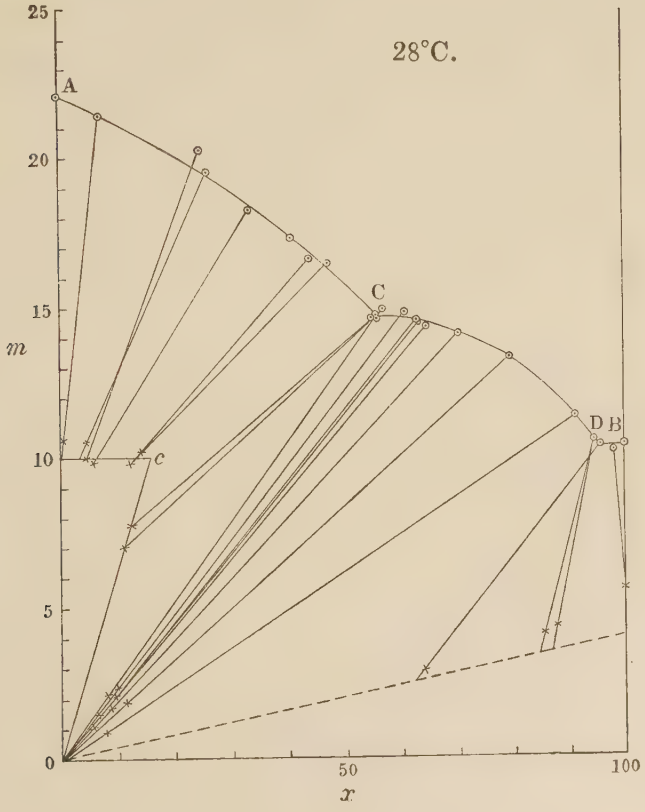


TABLE II.

Temperature : 31·0°.

No.	Solution.		Residue.	
	x	m	x	m
1	0·00	17·88	0·00	10·00
2	3·81	17·51	0·49	10·43
3	8·46	16·98	1·23	10·48
4	10.39	16·72	1·59	10·22
5	13·67	16·33	2·89	10·14
6	17·12	16·07	3·34	10·30
7	17·46	15·99	—	—
8	18·60	15·90	3·87	9·54
9	21.31	15·70	4·11	9·51
10	27·02	15·64	3·16	1·83
11	29·18	15·77	3·57	1·82
12	37·99	15·90	4·85	1·96
13	51·05	15·19	3·90	0·90
14	76.90	13·66	10·77	1·91
15	88·14	12·24	9·28	1·22
16	95·51	9·85	89·29	4·16
17	96·35	9·86	99·02	4·47
18	93·55	9·80	95·83	4·30
19	100·00	10·24	100·00	4·22

Fig. 2.

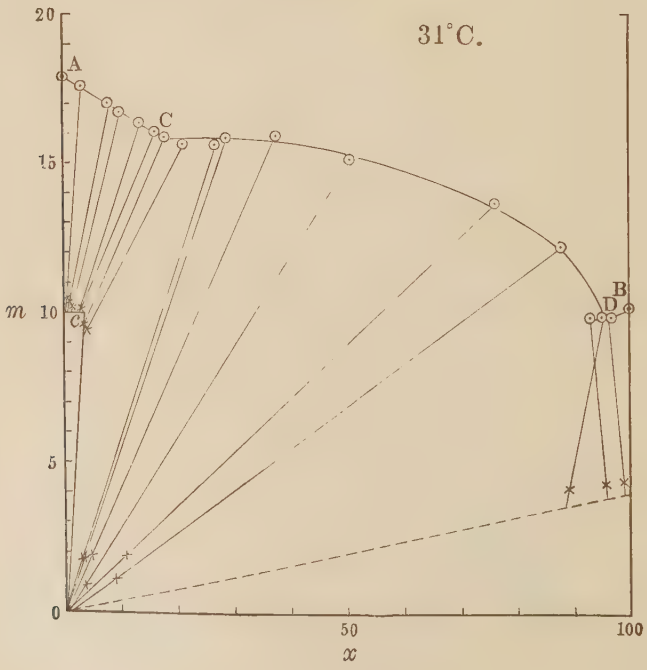
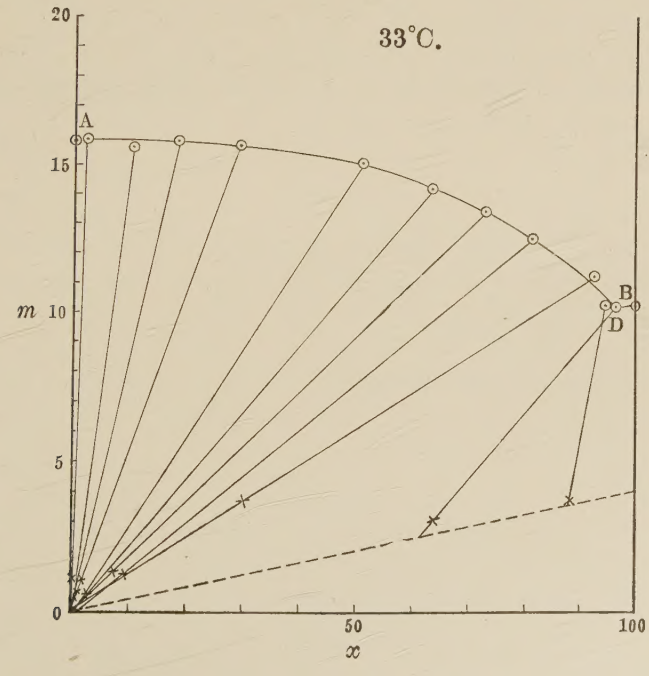


TABLE III.

Temperature: 33·0°.

No.	Solution.		Residue.	
	x	m	x	m
1	0·00	15·83	0·00	0·00
2	2·55	15·92	0·19	1·08
3	10·70	15·59	0·22	0·10
4	18·64	15·77	0·67	0·52
5	29·40	15·59	1·28	0·63
6	51·53	15·04	2·71	0·64
7	63·64	14·14	3·06	0·61
8	73·55	13·48	7·29	1·42
9	83·38	12·47	9·18	1·32
10	92·42	11·24	30·69	3·16
11	94·28	10·23	87·94	3·71
12	96·37	10·20	63·61	3·10
13	100·00	10·25	100·00	4·52

Fig. 3.



At 28° and 31°, the chromate is tetrahydrated and the solubility of the chromate in the sulphate as the decahydrate decreases as the temperature rises. It is estimated from the diagrams that the molar fractions of the chromate in the saturated solid solutions are 0·16 at 28° and 0·04 at 31°. Takeuchi noticed that the corresponding number at 25° is about 0·34 and it agrees approximately with the formula $2Na_2SO_4 \cdot Na_2CrO_4 \cdot 30H_2O$. It is certain, however that this is a mere accident and the saturated solid solution is no compound, as it may be seen from the above data at 28° and 31° and also from the experimental data, not published, that the molar fraction of the chromate in the saturated solid solution at 22° is ca. 0·8.

According to Th. W. Richards and W. B. Meldrum,[1] the tetrahydrate of sodium chromate takes up to a small extent the sulphate as tetrahydrate. Unfortunately, this interesting paper came to the notice of the authors when their work was ended, and as our results were insufficient to show the existence of those solid solutions, this fact was not taken into consideration in the diagrams and dotted lines were used for the unascertained parts.

The point, D, representing the solution saturated with the anhydrous sulphate and the tetrahydrated chromate (or that saturated with the sulphate), moves slightly to the right as the temperature rises.

At 33°, the sulphate is already anhydrous and no solid solution of the decahydrate exists, and as to the tetrahydrated chromate, what stated above applies in this case.

Summary.

The heterogeneous equilibria in the system of sodium sulphate, sodium chromate and water at 28°, 31° and 33° have been described and illustrated by diagrams.

[1] J. Amer. Chem. Soc., **43**, 1542 (1921).

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